Managing Public Portfolios*

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Abstract

We develop a unified framework for optimal management of public portfolios for a general class of macro-finance models imposing very few restrictions on households' risk and liquidity preferences or market structure for financial assets. Small-noise expansions to first-order conditions for a Ramsey plan can be reorganized into a formula for an optimal portfolio of government financial assets that isolates four motives balanced at an optimum: (1) hedging interest rate risk, (2) hedging primary deficit risk, (3) supplying liquid assets, and (4) internalizing equilibrium effects of public policies on financial asset prices. We directly calibrate quantitative measures of these four motives. Hedging interest rate risk plays a dominant role in shaping an optimal portfolio of financial assets for the U.S. federal government.

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1 Introduction

This paper isolates and quantifies motives that shape optimal government portfolios of financial assets. We do this for a class of representative household general equilibrium models. This class of models includes popular specifications of households' risk and liquidity preferences, sets of tradable securities, as well as restrictions that can limit access to some markets. These models have been used to explain asset price anomalies as well as price responses to changes in the supply of government debt. For this general class of models, we expand the "sufficient statistics" approach popular in public finance (see Chetty (2009)) and derive formulas for optimal portfolios that summarize the normative prescriptions using a small number of empirical moments. When applied to U.S. data, we find that the optimal debt portfolio is largely described by exponentially declining weights on longer maturities, and it needs little rebalancing over time.

Our framework consists of domestic households, foreign investors, and a benevolent government. Households are identical. They derive utility from consumption, leisure, and value liquidity services provided by a subset of securities. Our formulation of household preferences includes a variety of models studied in the literature – Ai and Bansal (2018) class of recursive preferences, discount factor shocks of Albuquerque, Eichenbaum, Luo, and Rebelo (2016), and also imperfect substitutability of financial assets in the spirit of Krishnamurthy and Vissing-Jorgensen (2012). Foreign investors are fully described by a set of demand functions for various securities. A benevolent government planner inherits representative agent's preferences and chooses a history contingent sequence of taxes and portfolios under commitment. We also allow for a variety of market structures by specifying which securities are traded by households, foreign investors and the government. Our market structures is general enough to encompass not only the classical complete markets model but also several types incomplete markets settings that have been used in applied work.

The planner faces costs from adjusting taxes and from altering the supply of securities that provide liquidity services, and internalizes the effects of its actions on asset prices. The planner uses returns on the securities it trades to smooth these costs across time and states. Private agents' first-order conditions for supplying labor and for purchasing securities traded by the government are "implementability" restrictions on allocations available to the planner. Combining these conditions with government budget constraints and small-noise expansions to the planner's first-order conditions yields a system of equations that determines the government portfolio.

The economic forces that drive the optimal composition are made transparent by decomposing the expression for the optimal portfolio in four intuitive terms: (i) *interest rate risk*, which arises from fluctuations in the current and future risk-free rates; (ii) *primary surplus risk* that arises from movements in the primary surplus; (iii) *liquidity risk* that captures the movements in households marginal benefits from liquidity services; and (iv) *price impact* that measures the effect on asset prices due to government's trades.

We show that all four terms are expressed in terms of a small number of moments that are easy to empirically measure. These sufficient statistics take the form of covariances such as the covariance of returns with each other, with interest rates, with deficits, and with liquidity premia, as well as elasticities such as the elasticity of tax revenues with respect to tax rates and elasticities of bond prices with respect to bond supply. Relative to the classical portfolio theory applied to the individual investor, neither risk aversion nor Sharpe ratios or betas appear in our formula. This is because the government is benevolent. To the extent it can trade the same securities as the households, its attitude to the risk-return trade-off must be the same as that of households, and these considerations disappear from its calculation of optimal portfolios.

The first term describes how the government can structure its portfolio to minimize risk from fluctuations in future interest rates. These movements are costly when the government needs to roll over its maturing debt. In general, the portfolio that hedges interest rate risk depends on the expected timing of deficits and how holding period returns comove with the yield curve. However, in a special case, when primitives are stationary and the government trades zero-coupon bonds, we show that the portfolio that hedges interest rate risk is straightforward. The government allocates a geometrically declining share of its portfolio in debts of longer maturities. Moreover, these weights only depend on the long-run average of the rate of return minus the long-run growth rate; so, it requires no rebalancing in response to temporary fluctuations.

The extent to which the government should depart from full hedging of interest rate risk depends on how well government bonds can hedge movements in primary deficits and liquidity. The second and third terms capture these considerations. Movements in primary surplus arise because of changes in tax rates and because of shocks that are exogenous to policy. We show that the portfolio that hedges primary surplus risk is determined by the covariance of holding period returns with the variation in primary deficits independent of fiscal policy.

Households value liquidity services, and therefore, increasing the supply of a security which has high marginal benefits from liquidity is welfare improving. However, as with taxes, movements in the marginal benefits of liquidity are costly. We show that the liquidity risk term is determined by a measure of excess liquidity premium and how holding period returns covary with the liquidity premia on short-maturity debt. Moreover, our formula says that the covariances of primary surplus with returns and covariances of liquidity premium with returns need to be scaled by the inverse covariances of holding period returns.

We use U.S. data to quantify the above-mentioned covariances using a parsimonious factor structure. We find that the shape of the overall portfolio is largely driven by the motive to hedge interest rate risk. These findings reflect the patterns in the U.S. data that covariances of deficits (or liquidity premia) with returns are small relative to the covariances of returns with each other. Furthermore, the portfolio that hedges primary surplus risk largely offsets the portfolio that hedges liquidity risk. Thus, U.S. debts appear to be a poor hedge for primary deficit and liquidity risks. Compared to observed U.S. debt portfolios, we find the optimal portfolio has a similar shape but a much longer duration. In addition to its geometrically-declining-weights shape, we also find that the optimal portfolio requires little rebalancing over time.

A second difference from classical portfolio theory is that the government, unlike a private investor, internalizes the effects of its trades on asset prices. The more the planner has to rebalance a price sensitive security, the more it needs to adjust taxes to raise the same amount of resources. We show that the these considerations are captured by the elasticities of bond prices to supply; and they have been extensively measured by segmented markets literature using evidence from quantitative easing policies. Using these estimates, we find that that the presence of price impact matters along a transition path but has a relatively small bearing on the stationary portfolio. The reason for this goes back to our previous finding that portfolio is largely shaped by by interest rate risk considerations and the portfolio that hedges interest rate risk requires little rebalancing.

Our findings contrasts to a large macro literature on optimal term structure of government debt that goes back to the seminal work of Angeletos (2002) and Buera and Nicolini (2004).[1] In studying a canonical neoclassical growth model, a typical finding in that literature is that in the government should issue long-term debt valued at tens or even hundreds times GDP while simultaneously taking an offsetting short position in short-term debt of a similar magnitudes. The optimal portfolio massively rebalances after aggregate shocks. Furthermore, the composition of an optimal portfolio is very sensitive to the menu of traded maturities. In contrast, we find moderate portfolios which are fairly stable over time. We show that the difference in findings is driven by counterfactual implications of the neoclassical growth model regarding the behavior of holding period returns on government debts. Standard parameterizations im-

[1] Other examples of such findings are in Farhi (2010); Faraglia, Marcet, Oikonomou, and Scott (2018); Lustig, Sleet, and Yeltekin (2008); Debortoli, Nunes, and Yared (2017)

ply that such returns are very smooth and highly correlated with fluctuations in the primary deficit, and as a result also with each other across maturities. This allows the government to hedge its shocks very well but it needs to take extreme debt positions to do so. Viewed through the lenses of our formula, such models imply that the primary surplus risk hedging term is very large and time-varying. In contrast, we show that in the U.S. data it is very small and stable.

2 Baseline environment

Timing and shocks. Time is discrete and infinite. Exogenous disturbances in period t are summarized by state $s_t \in \mathcal{S} \subset \mathbb{R}^S$. We assume that the state space $\mathcal{S}$ is compact, countable but can be finite or infinite. The initial state s_0 is predetermined. History of shocks is $s^t = (s_0, ..., s_t)$. We use $\Pr(s_t)$ and $\Pr\left(s_t | s^T\right)$ for $t > T$ to denote probabilities of s_t conditional on information in period 0 and s_T respectively. Similarly, we use $\Pr\left(s^t | s^T\right)$ for $t > T$ for the probability of s^t occurring conditional on s^T, with convention that $\Pr\left(s^t | s^T\right) = 0$ if s^t does not contain s^T. We write $s^t \succ s^T$ if s^t contains s^T. A value of variable x in state s^t is denoted by $x_t\left(s^t\right)$ or simply x_t if it is clear from the context what s^t we refer to. Similarly, we use interchangeably notation $\mathbb{E}_{s^t} x_{t+k}$ or $\mathbb{E}_t x_{t+s}$ for conditional expectation $\sum_{s^{t+k}} \Pr\left(s^{t+k} | s^t\right) x_{t+k}\left(s^{t+k}\right)$.

Securities. We impose minimal structure on asset markets. To make government portfolio problem interesting, we assume that there exist at least two securities, but the total number of securities is arbitrary otherwise and may be finite or infinite. A security i is characterized by an exogenous stream of payments $\left\{d_t^i\right\}_t$, which can be deterministic or stochastic; the set of states in which it can be traded; and the set of economic agents who can trade that security. Without loss of generality, we assume that $d_t^i\left(s^t\right) \subset s_t$ for all i, that is that state s_t includes realizations of payments for all securities. The net supply of security i in period t is denoted by $\mathcal{B}_t^i$, and it can be deterministic or stochastic.

The only restriction we impose on the market structure is that there exists a *one period government bond*, that we denote with superscript rf. This security is a pure discount bond issued by the government in period t that pays $d_{t+1}^{rf} = 1$ in all states in period $t+1$.

Price of security i is denoted by q_t^i. If security i cannot be traded by any agent in that period, we set $q_t^i = 0$. The *return* of security i, that can be traded in period $t-1$, is defined by $R_t^i \equiv \left(d_t^i + q_t^i\right) / q_{t-1}^i$. *Excess return* is defined as $r_t^i \equiv R_t^i - R_t^{rf}$, where R_t^{rf} is the return on a one period government bond issued in period $t-1$. Note that the definition of government bond implies that $R_t^{rf} = 1/q_{t-1}^{rf}$, so that R_t^{rf} is known at period $t-1$.

Economic agents. There are three types of economic agents: the government, households, and foreign investors. The *government* needs to finance an exogenous stream of expenditures G_t. To this end, it collects tax revenues and trades securities. To collect tax revenues, it imposes a proportional tax τ_t on output Y_t. Government's holdings of securities are denoted by $\left\{B_t^i\right\}_i$. We write the government budget constraint as

$$G_t + \sum_i q_t^i B_t^i = \tau_t Y_t + \sum_i \left(q_t^i + d_t^i\right) B_{t-1}^i. \tag{1}$$

To simplify notation, we do not distinguish explicitly in equation (1) between securities that the government can and cannot hold period t and simply sum over all securities i. Implicitly, we set $B_t^i = 0$ for all securities that the government cannot hold in period t. Initial portfolio of government securities is $\left\{B^i\right\}_{-1}$. We use $X_t \equiv G_t - \tau_t Y_t$ to denote the primary deficit.

There is a unit measure of identical *households*. Each household produces output y_t, pays taxes, trades securities, and consumes consumption good c_t. Household's budget constraint is

$$c_t + \sum_i q_t^i b_t^i = \left(1 - \tau_t\right) y_t + \sum_i \left(q_t^i + d_t^i\right) b_{t-1}^i. \tag{2}$$

Household preferences in period are defined recursively via

$$V_t = U_t\left(c_t, y_t, \left\{q_t^i b_t^i\right\}_i\right) + \beta \mathbb{W}_t\left(V_{t+1}\right), \tag{3}$$

where U_t is the utility function that may depend on s_t and $\mathbb{W}_t$ is a functional that maps $t+1$ measurable random variables to real numbers. We assume that U_t is twice continuously differentiable in all arguments, strictly increasing in c_t and decreasing in y_t; $\mathbb{W}_t$ is twice continuously differentiable and strictly increasing, increasing in first and second order stochastic dominance,[2] $\mathbb{W}_t(X) = X$ for any time-t measurable random variable. Households choose $\left(\mathbf{c}, \mathbf{y}, \left\{\mathbf{b}^i\right\}_i\right)$ to solve

$$\max_{\mathbf{c}, \mathbf{y}, \left\{\mathbf{b}^i\right\}_i} V_0 \tag{4}$$

subject to (2) and the initial conditions $\left\{b_{-1}^i\right\}_i$. We use $\beta^t \Pr\left(s^t\right) M_t\left(s^t\right)$ to denote the Lagrange multiplier on budget constraint (2) in state s^t.

This specification of household problem includes a variety of models of asset pricing and bond demands considered in the literature. The functional $\mathbb{W}_t$ is taken from work of Ai and

[2] In other words, $\mathbb{W}_t\left(X_{t+1}^1\right) \geq \mathbb{W}_t\left(X_{t+1}^2\right)$ whenever random variable X_{t+1}^1 first- or second-order stochastically dominates X_{t+1}^2.

Bansal (2018) who show that it incorporates a wide variety of models considered in the asset pricing literature: the recursive utility of Kreps and Porteus (1978) and Epstein and Zin (1989); the variational preferences of Maccheroni, Marinacci, and Rustichini (2006a), Maccheroni, Marinacci, and Rustichini (2006b); the multiplier preferences of Hansen and Sargent (2008) and Strzalecki (2011); the second-order expected utility of Ergin and Gul (2009); the smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (2005), Klibanoff, Marinacci, and Mukerji (2009); the disappointment aversion preference of Gul (1991); the recursive smooth ambiguity preference of Hayashi and Miao (2011). Moreover, by relaxing the differentiability assumption on $\mathbb{W}_t$, one can extend them to the maxmin expected utility of Gilboa and Schmeidler (1989), Epstein and Schneider (2003).

Similarly, specifications of U_t allows for both preference shocks in the spirit of Albuquerque, Eichenbaum, Luo, and Rebelo (2016) or imperfect substitutability of financial assets in the spirit of Krishnamurthy and Vissing-Jorgensen (2012). "Securities in the utility function" specification of U_t can be interpreted as an indirect utility resulting from frictions in asset markets. Suppose, for example, that the primitive utility function of the household $\tilde{U}_t(c_t, y_t)$ does not depend on securities directly, but households cannot trade some security j. The households maximize their preferences defined by utility $\tilde{U}_t$ function subject to the budget constraint (1) and an additional constraint $q_t^j b_t^j = 0$. This problem equivalently can be written as maximizing preferences defined by utility function $U_t \equiv \tilde{U}_t + \eta_t q_t^i b_t^i$ subject to the budget constraint (1), where η_t is proportional to the Lagrange multiplier this additional no-trade constraint.[3] In a similar way one can incorporate borrowing constraints, "bonds-in-advance" liquidity services provided by government-issued securities, etc.

Our analysis below will be substantially simplified if we abstract from income effects on labor supply. To this end, we assume that utility function can be represented as

$$U_t = U_t\left(c_t - \frac{(y_t/\theta_t)^{1+1/\gamma_t}}{1+1/\gamma_t}, \{q_t^i b_t^i\}_i\right), \tag{5}$$

where θ_t and γ_t are some (potentially stochastic) positive variables bounded away from zero. Using the consumption-leisure optimal choice of the household, we can represent a household's pre-tax earnings as

$$\ln y_t = \gamma_t \ln(1-\tau_t) + (1+\gamma_t)\ln\theta_t.$$

[3] To be able to write U_t in this form, the multiplier on constraint $q_t^j b_t^j = 0$ should be defined as $\beta^t \eta_t(s^t) \frac{\partial V_0}{\partial V_1(s^1)} \times ... \times \frac{\partial V_{t-1}(s^{t-1})}{\partial V_t(s^t)}$, where $\frac{\partial V_0}{\partial V_1(s^1)} \times ... \times \frac{\partial V_{t-1}(s^{t-1})}{\partial V_t(s^t)}$ is evaluated at the optimum. With standard time separable preferences, we simply have $\frac{\partial V_0}{\partial V_1(s^1)} \times ... \times \frac{\partial V_{t-1}(s^{t-1})}{\partial V_t(s^t)} = \Pr(s^t)$.

Thus, households earnings are function of tax rate τ_t, earning elasticity γ_t, and all other shocks that are captured by variable θ_t.

Finally, the *foreign investors* are specified by a set of time-t measurable demand functions $D_t^i\left(\left\{\mathbf{q}^i\right\}_i\right)$ for each security i. These functions are twice continuously differentiable, and can be stochastic and depend on s_t. This specification incorporates a variety of different specifications, such as closed economy (D_t^i are inelastic with $D_t^i = 0$ for all i, t), small open economy (D_t^i are perfectly elastic), noise traders in the spirit of Kyle (1985), or segmented markets in the spirit of Greenwood and Vayanos (2014).

Definition 1. *For given initial conditions $\left\{b_{-1}^i, B_{-1}^i\right\}_i$, a competitive equilibrium is a collection $\left(\boldsymbol{\tau}, \mathbf{c}, \mathbf{y}, \mathbf{Y}, \left\{\mathbf{b}^i, \mathbf{B}^i, \mathbf{q}^i\right\}_i\right)$ such that (i) $\left(\mathbf{c}, \mathbf{y}, \left\{\mathbf{b}^i\right\}_i\right)$ solves (4), (ii) $\left(\boldsymbol{\tau}, \mathbf{Y}, \left\{\mathbf{q}^i, \mathbf{B}^i\right\}_i\right)$ satisfies (1), (iii) $\mathbf{y} = \mathbf{Y}$ and $\mathbf{b}^i + \mathbf{B}^i + \mathbf{D}^i = \boldsymbol{\mathcal{B}}^i$ for all i.*

3 Optimal public portfolios

Our paper focuses on the analysis of the optimal structure of government portfolio $\left\{\mathbf{B}_t^i\right\}_i$ chosen by a benevolent government planner under commitment. Thus, the planner maximizes the household utility V_0 and chooses policy variables $\left(\boldsymbol{\tau}, \left\{\mathbf{B}_t^i\right\}_i\right)$ such that $\left(\boldsymbol{\tau}, \mathbf{c}, \mathbf{y}, \mathbf{Y}, \left\{\mathbf{b}^i, \mathbf{B}^i, \mathbf{q}^i\right\}_i\right)$ is a competitive equilibrium. In this section, we use small-noise approximations to the planner's optimality conditions to characterize the optimal portfolio.

3.1 Perturbations and approximations

Before we go into specific analysis, it is useful to give a broad overview of our approach. Take any competitive equilibrium $\left(\boldsymbol{\tau}, \mathbf{c}, \mathbf{y}, \mathbf{Y}, \left\{\mathbf{b}^i, \mathbf{B}^i, \mathbf{q}^i\right\}_i\right)$. Suppose that the government decides to slightly perturb its portfolio of securities after some history. For this perturbation to be feasible, that is, satisfy budget constraints, the government would need to adjust taxes $\boldsymbol{\tau}$ as well. We parameterize the size of this perturbation by parameter ϵ and use notation $\left(\boldsymbol{\tau}_\epsilon, \mathbf{c}_\epsilon, \mathbf{y}_\epsilon, \mathbf{Y}_\epsilon, \left\{\mathbf{b}_\epsilon^i, \mathbf{B}_\epsilon^i, \mathbf{q}_\epsilon^i\right\}_i\right)$ to denote the competitive equilibrium under the perturbed policy. For any equilibrium variable x_t we use notation $\partial_\epsilon x_t$ denote the derivative $\partial_\epsilon x_t \equiv \lim_{\epsilon \to 0} \left(x_{t,\epsilon} - x_t\right)/\epsilon$.

Welfare effect from this perturbation as the size of the perturbation goes to zero is, due to the envelope theorem, given by

$$\partial_\epsilon V_0 = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t M_t\left(s^t\right)\left[-Y_t\left(s^t\right)\partial_\epsilon\tau_t\left(s^t\right)-\sum_i\left(b_t^i\left(s^t\right)-b_{t-1}^i\left(s^{t-1}\right)\right)\partial_\epsilon q_t^i\left(s^t\right)\right] \tag{6}$$
$$+\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\frac{\partial V_0}{\partial V_1\left(s^1\right)}\times ...\times\frac{\partial V_{t-1}\left(s^{t-1}\right)}{\partial V_t\left(s^t\right)}\frac{\partial U_t\left(s^t\right)}{\partial\left(q_t^i b_t^i\right)}b_t^i\left(s^t\right)\partial_\epsilon q_t^i\left(s^t\right).$$

The main take away from this equation is that the welfare effect of the perturbation depends only on objects that are directly known in equilibrium, such as $\left(\mathbf{M},\mathbf{Y},\left\{\mathbf{b}^i\right\}_i\right)$, and on tax and price responses $\partial_\epsilon\boldsymbol{\tau}$ and $\left\{\partial_\epsilon\mathbf{q}^i\right\}_i$. This significantly simplifies our analysis. As long as such tax and price responses can be inferred from the data, one does not need to take a strong stand on the specific functional forms for household's preferences or a specific mechanism that determines asset prices.

If the government's portfolio in the competitive equilibrium is optimal, then this perturbation cannot increase welfare, so we should have $\partial_\epsilon V_0 \leq 0$. By considering the opposite perturbation with parameter $-\epsilon$, we then establish that optimality requires

$$\partial_\epsilon V_0 = 0. \tag{7}$$

To connect this condition back to the optimal government portfolio we use the government's budget. Applying the perturbation to the budget constraint yields

$$\partial_\epsilon\left(\tau_t Y_t\right)-\sum_i\partial_\epsilon q_t^i\left(B_t^i-B_{t-1}^i\right)=\sum_i q_t^i\partial_\epsilon B_t^i-\sum_i\left(q_t^i+d_t^i\right)\partial_\epsilon B_{t-1}^i. \tag{8}$$

Combining budget identity (8) with the optimality condition (7) one can establish conditions that the optimal portfolio must satisfy.

There are two difficulties that we need to overcome to make this approach operational. The first difficulty comes from the fact that the mapping between the optimality condition (7) and the government budget constraint (8) is, in general, non-linear and complicated. The second challenge is that responses $\partial_\epsilon\boldsymbol{\tau}$ and $\left\{\partial_\epsilon\mathbf{q}^i\right\}_i$ would depend on a specific perturbation ϵ one considers. Since there are infinite number of different perturbations, in principle, one would need to know infinite number of possible tax and price responses. This is not very practical.

We overcome both challenges by developing a particular class of second-order expansions of equilibrium variables. Our approximations techniques build on the ideas used in asset pricing literature and computational economics, such as Samuelson (1970), Devereux and Sutherland (2011), Schmitt-Grohe and Uribe (2004), Bhandari, Evans, Golosov, and Sargent (2021). Fix any state s^T. Without loss of generality, we can write vector $s^t \geq s^T$ as

$$s_t = \mathbb{E}_T s_t + \varepsilon_t \equiv \bar{s}_t + \varepsilon_t,$$

where $\mathbb{E}_T \varepsilon_t = 0$. Consider a sequence of stochastic processes, parameterized by scalar $\sigma \geq 0$, defined as

$$s_t(\sigma) = \bar{s}_t + \sigma \varepsilon_t.$$

Here $\sigma = 0$ corresponds to a deterministic economy in which all uncertainty is shut down after state s^T. Let $x_t(\sigma)$ be any equilibrium variable in the σ-economy. We use second order Taylor expansions of equilibrium conditions with respect to σ around $\sigma = 0$ and use "$\simeq$" sign to denote any relationship that hold with equality up to the order $O\left(\sigma^3\right)$. We use $\bar{x}_t, \partial_\sigma x_t, \partial_{\sigma\sigma} x_t$ to denote zeroth, first and second order terms in expansion, so that in this notation

$$x_t(\sigma) \simeq \bar{x}_t + \partial_\sigma x_t + \frac{1}{2}\partial_{\sigma\sigma} x_t.$$

Implicitly, throughout our analysis we assume that equilibrium is sufficiently well-behaved. In particular, we assume that perturbations ϵ, and $-\epsilon$ are feasible for small ϵ; there is a unique equilibrium for each ϵ; and the equilibrium manifold is smooth, so that the limits as $\epsilon \to 0$ are well defined. Similarly, we assume that equilibrium is smooth and unique in σ for small values of this parameters, and that present value of government constraint at each state is finite. We call such economies *regular*. While it would be interesting to explore sufficient conditions for the existence of regular equilibria, that would require imposing additional structure on model's primitives that would distract from the main focus of the paper, and therefore we leave such extensions to future work.

3.2 Tax revenue elasticities and liquidity premium

There are two objects that will appear frequently in our analysis. The first one is *tax revenue elasticity*, that we denote by ξ_t, and define by

$$\xi_t \equiv \frac{\partial \ln(\tau_t Y_t)}{\partial \ln \tau_t} = 1 - \gamma_t \frac{\tau_t}{1-\tau_t}.$$

It provides a measure of tax distortions. When $\xi_t = 1$, the output does not respond to tax rates and there is no distortions. More generally, $1 - \xi_t$ measures the deadweight losses from transferring resources between the government and the households. For our analysis, we will often require to know by how much the government needs to increase (or decrease) tax rates in order to raise (or return) 1 unit of resources for households. It is easy to verify that this value is given by $\frac{1}{\xi_t Y_t}$.

The second object we will use is what we call *liquidity premium* or *liquidity wedge* for security i, that we denote by a_t^i and define by

$$1 - a_t^i \equiv \mathbb{E}_t \frac{\beta M_{t+1}}{M_t} R_{t+1}^i. \tag{9}$$

To get an intuition for this definition, consider first any security that households can trade freely and that does not give them any direct utility benefit. The consumer optimality condition with respect to this security is $1 = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t} R^i_{t+1}$ and therefore the liquidity premium is zero. Therefore, a^i_t for any security i is a measure of additional benefits or costs that this security carries beyond transferring resources between periods. In the data, returns on government-issued debt is often lower than returns on debt issues by private sector, and one common explanation for this phenomenon is government-issued bonds provides additional liquidity services. Through the lens of our definition, under this explanation private debt carries no liquidity premium, while public debt has positive liquidity premium.

4 Optimal public portfolios in a small open economy

As equation (7) makes it clear, two types of responses play an important role in determining optimal policy, the tax response $\partial_\epsilon \boldsymbol{\tau}$ and the price response $\left\{\partial_\epsilon \mathbf{q}^i\right\}_i$. It will be instructive to consider first the case in which price response is always zero, as would be in the case of a small open economy. Many techniques and insights developed in this case will continue to hold more generally, but the arguments are be simpler and more transparent.

To build the intuition behind forces that determine optimal public portfolio, it is useful to start with the following thought experiment. Suppose that in period T, in some state s^T, the government reduces tax rates to lower its revenues by ϵ dollars. To offset the revenue fall, the government also sells ϵ dollars of security j. In period $T+1$ it buys back the same quantity of security that it sold in previous period, and adjusts tax rates to satisfy its budget constraint in period $T+1$.

Consider the implication of this transaction on tax rates for a small ϵ. Reducing revenues by ϵ dollars requires lowering taxes by $\frac{\epsilon}{Y_T \xi_T}$ and selling $\frac{\epsilon}{q^j_T}$ units of security j. Each unit of bond sold has d^j_{T+1} opportunity cost in terms of dividends and q^j_{T+1} units of resources required to buy it back in period $T+1$. Therefore, in period $T+1$ taxes need to be increased to raise $\frac{q^j_{T+1}+d^j_{T+1}}{q^j_T}\epsilon$ units of resources or tax rates would need to increase by $R^j_{T+1}\frac{\epsilon}{Y_{T+1}\xi_{T+1}}$. No additional tax adjustments are needed in any other period and, therefore, the welfare impact of this perturbation is

$$\begin{aligned} \frac{\partial_\epsilon V_0}{\beta^T \Pr\left(s^T\right) M_T\left(s^T\right)} &= \frac{1}{\xi_T} - \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} R^j_{T+1} \frac{1}{\xi_{T+1}} \\ &= \left\{\left(\frac{1}{\xi_T}-1\right) - \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} R^j_{T+1}\left(\frac{1}{\xi_{T+1}}-1\right)\right\} + a^j_T, \end{aligned} \tag{10}$$

where in the second line we use the definition of liquidity wedge (9). All period T variables on the right hand size are a function of s^T that we omitted for brevity.

The first line show that welfare impact of this transaction is intimitately tied to the tax revenue elasticity. This transaction transfers 1 unit of resources from period $T+1$ to period T with a stochastic return R^j_{T+1}. Increase in welfare in period T from more resources is given by $\frac{1}{\xi_T}$, while decrease in period $T+1$ is given by $R^j_{T+1}\frac{1}{\xi_{T+1}}$. To discount the future uncertain reduction of resources, the benevolent government uses households shadow stochastic discount factor $\frac{\beta M_{T+1}}{M_T}$.

The second equation separates this welfare effect into welfare impact from deadweight losses $\left(\frac{1}{\xi_t}-1\right)$ and from liquidity premium a^j_t. For concreteness, suppose that security j is a government bond (so that selling it means that the government issues more debt) and that it has positive liquidity premium, $a^j_T>0$. If taxes are not distortionary, or when $\xi_t=1$ for all t, then $\frac{\partial_\epsilon V_0}{\beta^T \Pr(s^T) M_T(s^T)} = a^j_T > 0$, so issuing more debt with positive liquidity premium is welfare improving. In the optimum, the government should be issuing enough debt to satiate household's demand for it and bring the liqudity premium to zero. When taxes are distortionary, issuing more debt in period T entails lower tax rates and deadweight losses $\left(\frac{1}{\xi_T}-1\right)$, and higher deadweight losses $R^j_{T+1}\left(\frac{1}{\xi_{T+1}}-1\right)$ in period $T+1$. The first two terms in the second line of equation (10) shows these two effects.

The focus of our analysis is on the optimal composition of the portfolio of government's securities holdings. Such portolio can be analyzed by considering a transaction in which the government sells security j in any period T and simultaneously conducts the opposite offsetting transaction with another security. Without loss of generality, we can set the offsetting security to be the one period government bond. In portfolio is optimal, neither such transaction, nor the opposite transaction that buys security j financed by selling a one period bond can increase welfare. Therefore, combining equation (10) for security j and for rf, and setting the net welfare effect to zero, we obtain

$$a^j_T - a^{rf}_T = cov_T\left(\frac{\beta M_{T+1}}{M_T} r^i_{T+1}, \frac{\left(\xi_{T+1}\right)^{-1}}{\mathbb{E}_T\left(\xi_{T+1}\right)^{-1}}\right).$$

To the second order of approximation, this can be written as (see appendix)

$$cov_T\left(\ln \xi_{T+1}, r^j_{T+1}\right) \simeq \frac{a^{rf}_T - a^j_T}{1-a^{rf}_T} R^{rf}_{T+1}. \tag{11}$$

It is instructive to compare this formula to its analogue in the classical portfolio theory applied to individual investor. In the classical portfolio theory, the investor chooses portfolios so that covariance of excess returns with her labor earnings is equalized to some measure of covariance of those returns with broad market index adjusted by investor's risk aversion (see, e.g. Viceira (2001)). The analogue of investor's labor earnings in our problem are deadweight losses ξ_{T+1}, but that's where the similarly ends between the two formulas. Neither risk aversion, nor assets Sharpe ratios or betas appear in formula (11). Instead, the covariance of deadweight losses and returns is equalized to a measure that is proportional to the excess liquidity premium that asset j pays over the government risk free bond, $a_T^j - a_T^{rf}$.

To understand the intuition for this result, it is useful to first consider the case when households can freely trade both assets without any direct utility benefits, so that their liquidity premia are zero. In this case, equation (11) implies that the government wants to choose a portfolio that hedgees fluctuation in deadweight losses by minimizing co-movement between ξ_{T+1} and r_{T+1}^j. There is no traditional "risk-return" consideration, captured by risk aversion and covariance of returns for asset j with the broad market index, that are central to the classical portfolio theory. Since the government is benevolent and can trade the same assets that households do, its attitude to the risk-return trade-off must be the same as that of households. Trading an asset with them cannot increase welfare simply because its return appears to be high relative to risk, since that requires households to hold the opposite view to be willing to be a counter-party for government's transaction.

This logic breaks down if households cannot trade these two assets, or when they provide additional liquidity benefits. Equation (11) shows that the government wants to equalize covariance of deadweight losses and excess returns to a measure of liquidity wedge, given by the right hand side of (11). If taxes are not distortionary, then $\ln \xi_t = 0$ for all t, in which case optimal portfolio simply eliminates excess liquidity premium on all assets the government can trade. When taxes are distortionary, this consideration is offset by additional risk to deadweight losses that such assets carry.

Finally, consider a generalization of the perturbation considered so far. Suppose that the government rolls over, using the one period bond, the excess return it obtained in period $T+1$ for k more periods and then adjusts taxes in period $T+1+k$ to return (or finance) these resources. Using the same arguments as before, it is easy to obtain a generalization of (11)

$$cov_T\left(\ln \xi_{T+1+k}, r_{T+1}^j\right) + cov_T\left(\mathrm{A}_{T+1}^k, r_{T+1}^j\right) \simeq \frac{R_{T+1}^{rf}}{1-a_T^{rf}}\left(a_T^{rf} - a_T^j\right), \tag{12}$$

where $\mathrm{A}_{T+1}^k \equiv \sum_{t=1}^k a_{T+t}^{rf}$ is the liquidity premium on the risk-free bond accumulated between period $T+1$ and $T+1+k$.

We now derive implication of this equation for the optimal composition of government portfolio. Consider the present value of government budget constraint in period $T+1$. It can be written as

$$\mathbb{E}_{T+1}\sum_{t=1}^{\infty}\mathcal{Q}_{T+1}^{T+t}X_{T+t}=B_T\left[R_{T+1}^{rf}+\sum_{i\geq 1}\omega_T^i r_{T+1}^i\right], \tag{13}$$

where

$$\mathcal{Q}_{T+1}^{T+t}\equiv 1\times\frac{1}{\sum_{i\geq 1}r_{T+2}^i\omega_{T+1}^i+R_{T+2}^{rf}}\times...\times\frac{1}{\sum_{i\geq 1}r_{T+t}^i\omega_{T+t-1}^i+R_{T+t}^{rf}}$$

are discount rates between period $T+1$ and $T+t$, and $\sum_{i\geq 1}$ denotes a sum over all assets $i\neq fr$. One can show that to the first order approximation expected excess returns must be zero, so the first order approximation of the government budget constraint satisfies

$$\sum_{t=1}^{\infty}\overline{X}_{T+t}\mathbb{E}_{T+1}\partial_\sigma Q_{T+1}^{T+t}+\sum_{t=1}^{\infty}\overline{Q}_{T+1}^{T+t}\mathbb{E}_{T+1}\partial_\sigma X_{T+t}=\left(\sum_{i\geq 1}\partial_\sigma r_{T+1}^i\overline{\omega}_T^i\right)\overline{B}_T+\partial_\sigma\left(B_T R_{T+1}^{rf}\right), \tag{14}$$

where $Q_{T+1}^{T+1}=1$ and $Q_{T+1}^{T+t}\equiv 1\times q_{T+1}^{rf}\times...\times q_{T+t-1}^{rf}$ for $t>1$ is the discount rate calculated using one period government discount rates. At this point, this is still an identify. It says that the first order approximation of the budget constraint can be decomposed into fluctuations in interest rates (the first expression on the left hand side of equation (14)) and fluctuations of primary deficits (the second expression on the left hand size of (14)). These fluctuations should be equal to fluctuations in the value of government portfolio, the expression on the right hand size of (14). As long as the present value of the government budget constraint is finite, this equation holds in any equilibrium.

To obtain implications for the optimum portfolio, we substitute the optimality conditions (11) and (12). These conditions show the relationship between returns and tax revenue elasticities ξ_t. To connect it to the expressions that appear in the budget constraint (14), observe that owing to the assumption of no income effects, we can decompose fluctuations in primary deficit $\partial_\sigma X_t$ into fluctuations attributed to fluctuations in $\partial_\sigma\ln\xi_t$, and in fluctuations attributed to other shocks. In particular, we show in the appendix, that we can write

$$\partial_\sigma X_t=\bar{\zeta}_t\bar{Y}_t\partial_\sigma\ln\xi_t+\partial_\sigma X_t^{\perp}, \tag{15}$$

where $\zeta_t\equiv\xi_t^2\frac{(1-\tau_t)^2}{\gamma_t}$, $\bar{\zeta}_t$ is its zeroth order approximation, and $\partial_\sigma X_t^{\perp}$ is independent of $\partial_\sigma\tau_t$. We now apply this decomposition to (14), multiply both sides of that equation by $\partial_\sigma r_{T+1}^j$ and take expectations at time T, and substitute the optimality conditions (11) and (12) to characterize the optimal public portfolio.

We summarize the optimal portfolio in the following theorem. Let $i=1,...,I\leq\infty$ be the set of securities, in addition to the one period bond, that the government can trade and let $\boldsymbol{\omega}_T$ be a vector that summarizes the portfolio of those securities in period T with elements

$\boldsymbol{\omega}_T[i] = \omega_T^i$. Define matrices Σ_T^Q, Σ_T^X, Σ_T^a, Σ_T and diagonal matrices Π_T^Q, Π_T^a, Π_T^A with elements as shown in the following table.

$\Sigma_T^Q[j,t] = cov_T\left(\ln Q_{T+1}^{T+1+t}, r_{T+1}^j\right)$	$\Pi_T^Q[t,t] = \mathbb{E}_T \frac{q_{T+t}^{rf} X_{T+1+t}}{Y_{T+t}}$
$\Sigma_T^A[j,t] = cov_T\left(\mathrm{A}_{T+1}^t, r_{T+1}^j\right)$	$\Pi_T^A[t,t] = \mathbb{E}_T q_{T+t}^{rf} \frac{Y_{T+1+t}}{Y_{T+t}} \zeta_{T+1+t}$
$\Sigma_T^X[j,t] = cov_T\left(\frac{X_{T+t}^{\perp}}{\mathbb{E}_T Y_{T+t}}, r_{T+1}^j\right)$	$\Sigma_T[j,i] = cov_T\left(r_{T+1}^i, r_{T+1}^j\right)$
$\Pi_T^a[t,t] = \frac{\mathbb{E}_T \zeta_{T+t}}{\left(1-a_T^{rf}\right) q_T^{rf}}$	$\Sigma_T^a[j,t] = a_T^{rf} - a_T^j$

Finally, let $\mathbf{w}_T$ be a vector with elements $\mathbf{w}_T[t] = \mathbb{E}_T q_T^{rf} Q_{T+1}^{T+t} \frac{Y_{T+t}}{Y_T}$.

Theorem 1. *(i). The optimal portfolio $\boldsymbol{\omega}_T$ satisfies*

$$q_T^{rf} \frac{B_T}{Y_T} \Sigma_T \boldsymbol{\omega}_T \simeq \left[\Sigma_T^Q \Pi_T^Q + \Sigma_T^X + \left(\Sigma_T^a \Pi_T^a - \Sigma_T^A \Pi_T^A\right)\right] \mathbf{w}_T. \tag{16}$$

(ii). If, in addition, there some Γ, q such that the following stationary conditions are satisfied

$$\mathbb{E}_T \tau_{T+t} \approx \tau_T,\ \mathbb{E}_T \gamma_{T+t} \approx \gamma_T,\ \mathbb{E}_T \frac{X_{T+t}}{Y_{T+t}} \approx \frac{X_T}{Y_T}, \mathbb{E}_T \frac{Y_{T+t+1}}{Y_{T+t}} \approx \Gamma,\ \mathbb{E}_T q_{T+t}^{rf} \approx q, \tag{17}$$

where "≈" means that the relationship holds with equality up to order $O(\sigma)$, then the optimal portfolio $\boldsymbol{\omega}_T$ satisfies

$$q \frac{B_T}{Y_T} \Sigma_T \boldsymbol{\omega}_T \simeq \left[(1 - q\Gamma) \frac{B_T}{Y_T} \Sigma_T^Q + \Sigma_T^X - \zeta_T q \Gamma \Sigma_T^A\right] \mathbf{w}, \tag{18}$$

where $\mathbf{w}$ is a vector with elements $\mathbf{w}[t] = (q\Gamma)^t$.

Theorem 1(i) shows that the optimal public portfolio is determined by the need to hedge three sources of risk: interest rate risk, captured by matrix Σ_T^Q, risk to primary deficits, captured by Σ_T^X, and the liquidity risk, captured by Σ_T^a and Σ_T^A. The vector $\mathbf{w}_T$ shows how the government discounts different risks inter-temporally. It is equal to the discount rate on government bonds adjusted by the growth rate of the economy. Finally, matrices Π_T^Q, Π_T^a, and Π_T^A are adjustment factors that emerge if there are predictable variations in interest rates, growth rates, and taxes.

These expressions further simplify if economy is approximately stationary, in the formal sense defined by equation (17). In this case, adjustment matrices Π_T^Q, Π_T^a, and Π_T^A become constants that show the relative importances of the three types of risks for the government.

By observing equation (18), one can see that the interest rate risk directly scales of the amount of debt, $\frac{B_T}{Y_T}$. To understand why interest rate risk is increasing with debt, observe that the interest rate risk emerges because the government needs to adjust its portfolios in the future, for example, because it needs to roll over existing debt obligations that are coming due. The larger the outstanding debt obligations are, the costlier the interest rate risk is, and it plays a larger role in determining the optimal portfolio.

The primary deficit risk is determined by the covariance of primary deficits with returns (or, to be more precise, the covariance in primary deficits not associated with fluctuations in tax revenue elasticity). Liquidity risk is scaled by parameter ζ_T. Simple algebra shows that $\zeta_T = \frac{(1+\gamma_T)^2}{\gamma_T}\left(\frac{1}{1+\gamma_T} - \tau_T\right)^2$. ζ_T is decreasing in τ_T and reaches 0 at $\tau_T = \frac{1}{1+\gamma_T}$, which corresponds to the peak of the Laffer curve. To understand why current tax levels affect the importance of hedging interest risk, it is useful to consider the following thought experiment. Suppose that government can borrow at cheaper rate than household. Then the government can help households by borrowing on their behalf. The benefit from this transaction comes from lower interest rates that the government faces. The cost comes from distortions that arise from higher taxes that such borrowing must entail. The closer the taxes are to the peak of the Laffer curve, the larger are relative cost of tax distortions to benefits of liquidity provision.

4.1 Optimal portfolio of public debts

Theorem 1 does not take a stance on which securities the government can trade, and characterizes the optimal portfolio for an arbitrary set of such securities. The most common securities traded by the government are government debts of various maturities. In this section we explore the implications of theorem 1 for the optimal debt maturity.

We assume that the government debts come in the form of pure discount bonds (that is, a bond that has no coupon payments and pays 1 unit of resources at some specified maturity date) and that the government can issue debt of any maturity. For the purposes of applying theorem 1, security i will correspond to a bond that matures in period $T+1+i$. In the appendix we show the following corollary to theorem 1.

Corollary 2. *If government portfolio consists of pure discount bonds as described above, then* $q_T^{rf}\Sigma_T \simeq \Sigma_T^Q$. *In particular, under stationary condition (17), the optimal portfolio satisfies* $\boldsymbol{\omega}_T \approx \mathbf{w}_T^*$, *where*

$$\mathbf{w}_T^* \equiv (1-q\Gamma)\,\mathbf{w} + \left[\frac{Y_T}{qB_T}\Sigma_T^{-1}\left(\Sigma_T^X + \zeta_T\left(\frac{1}{q\left(1-a_T^{rf}\right)}\Sigma_T^a - (q\Gamma)\,\Sigma_T^A\right)\right)\right]\mathbf{w}. \tag{19}$$

One implication of the fact that $q_T^{rf}\Sigma_T \simeq \Sigma_T^Q$, is that the government can hedge the interest rate risk fully, at least to the order of approximation we consider. Recall that the interest rate risk emerges because the government needs to roll over its maturing debt. But if the government issues debt to match the amount that is due to the expected primary surplus in future periods then there is no need to roll over debt at all (at least, to the second order) and so the interest risk is eliminated.

The portfolio that fully hedges interest rate risk becomes especially simple in stationary economy. In this case, primary surpluses grow at a constant rate and hence the government chooses a portfolio for which the number of bonds maturing in each period grows with the same rate. The price of a pure discount bond that matures in period $T+t$ is approximately equal to q^t, and hence the market values of debts of different maturities form a geometric sequence: the fraction of market value of debt that matures in period $T+t$ in the total market value of government debt is equal to $(1-q\Gamma)(q\Gamma)^t$. This is the first term on the right hand side of equation (19).

How much the government should depart from full hedging of interest rate risk depends on how well government bonds can hedge primary deficit and liquidity risks. This is given by the expression in the square brackets in equation (19). In the next section, we will make an attempt to estimate the value of this expression using U.S. data and find that it is fairly small. Thus, U.S. debts appear to be a poor hedge for primary deficit and liquidity risks. This carries additional implications about which securities U.S. government should invest beyond issuing public debt. Since public debt can always hedge interest risk fully, the largest gains would arise from choosing securities that provide a good hedge against primary deficit and liquidity risks.

5 Quantifying the Optimal Portfolio

In this section, we use U.S. data to measure sufficient statistics that appear in a version of the formula developed in the previous section. To bring the formula to the data, we need to take a stand on the market structure, and impose some assumptions on the stochastic processes for asset prices, deficits, and so on. For the results below, we assume that the government trades zero-coupon non-state contingent bonds of all maturities. This serves a natural benchmark (see Angeletos, 2002, Buera and Nicolini, 2004), and well-approximates the portfolio for the U.S. To keep the number of objects to be estimated tractable, we start with a stronger form of stationarity on the stochastic processes than equation (17). In particular, we assume that the conditional means are time-independent. Under these assumptions, we will estimate all the terms that appear in (19) and then compare the optimal portfolio to the observed U.S. portfolio.

5.1 Data

We use quarterly data on prices of U.S. treasury and AAA bonds, a measure of tax rates, and primary deficits. Most of data spans the period $1952-2017$, with the exception of data on the prices of AAA bonds, for which data is only available after 1984. We first describe the construction of the main variables used in the analysis—excess returns on bonds of different maturities $\left\{r_t^j\right\}_{j,t}$, prices of AAA bond $\left\{q_t^{AAA,j}\right\}_{j,t}$, primary deficits $\{X_t\}_t$, and tax rates $\{\tau_t\}$. Then, we describe how these data maps to the objects we need to implement our optimal portfolio formula (19)—that is, covariances $\left\{\Sigma_T, \Sigma_T^X, \Sigma_T^A, \Sigma_T^a\right\}$ and constants $\{q, \Gamma, \zeta\}$.

Bond prices Our formulas require us to measure $\left\{r_t^j\right\}_{j,t}$, which are the excess holding period returns on bonds of all maturities. The holding period returns come from Fama Maturity Portfolios published by CRSP. There are 11 portfolios, each adjusted monthly, to hold bonds of maturities in specified interval– starting from maturities of 6 to 60 months in 6 month intervals, and a final portfolio for maturities between 60 and 120 months. For each portfolio, we take the center of the interval to which the portfolio's maturities corresponds as this portfolio's maturity. The holding period excess return equals the holding period return minus the nominal short rate, which is measured by the quarterly 3-Month Treasury Bill, and then we adjust the returns for movements in expected inflation. The plots of holding period excess returns are in figure 1 and the summary statistics is in table 1. The mean and the volatility of the excess holding period returns are increasing in the maturity. In figure 1, we see that there is a significant comovement across the returns.

As an input to measure the liquidity premium, we use data on prices of AAA securities. Our main source is Treasury.gov which computes High Quality Market (HQM) Corporate Bond Yield Curve for the Pension Protection Act and uses a methodology developed at US Treasury to construct corporate bond yield curves by using extended regressions on maturity ranges. The HQM yield curve represents the high quality corporate bond market, i.e., corporate bonds rated AAA, AA, or A.[4]

Deficits and Taxes Deficits are measured by the real federal government spending minus the real federal government revenues from the national income and product accounts. The real federal government spending is measured by the sum of "federal government consumption expenditures" and "federal government current transfer payments: government social benefits: to persons", deflated by the implicit price deflator for GDP. The real federal government

[4] For more background information see "https://www.treasury.gov/resource-center/economic-policy/corp-bond-yield/Pages/Corp-Yield-Bond-Curve-Papers.aspx."

Table 1: Summary Statistics for Real Holding Period Excess Returns

maturity	mean	std	min	25%	50%	75%	max
6	0.08	0.46	-2.37	-0.17	0.05	0.28	2.48
9	0.15	0.89	-3.68	-0.29	0.07	0.50	5.68
15	0.21	1.37	-6.00	-0.53	0.15	0.82	8.15
21	0.23	1.70	-7.18	-0.74	0.14	1.03	9.45
27	0.27	2.01	-7.81	-0.84	0.13	1.31	11.38
33	0.31	2.21	-8.50	-0.97	0.07	1.51	12.01
39	0.33	2.37	-9.68	-1.10	0.07	1.68	11.45
45	0.34	2.53	-10.13	-1.18	0.08	1.64	12.47
51	0.36	2.65	-11.10	-1.31	0.07	1.99	12.45
57	0.29	2.93	-11.19	-1.62	0.01	2.00	15.28
90	0.45	3.25	-12.36	-1.49	0.08	2.12	15.04

Notes: This table records the number of observations, mean, standard deviation, minimum value, 25th percentile, 50th percentile, 75 percentile and maximum value of the sample of real quarterly holding period excess returns for issues with maturities from 6 months to 90 months. The units of the returns are percents and the unit of maturity is month.

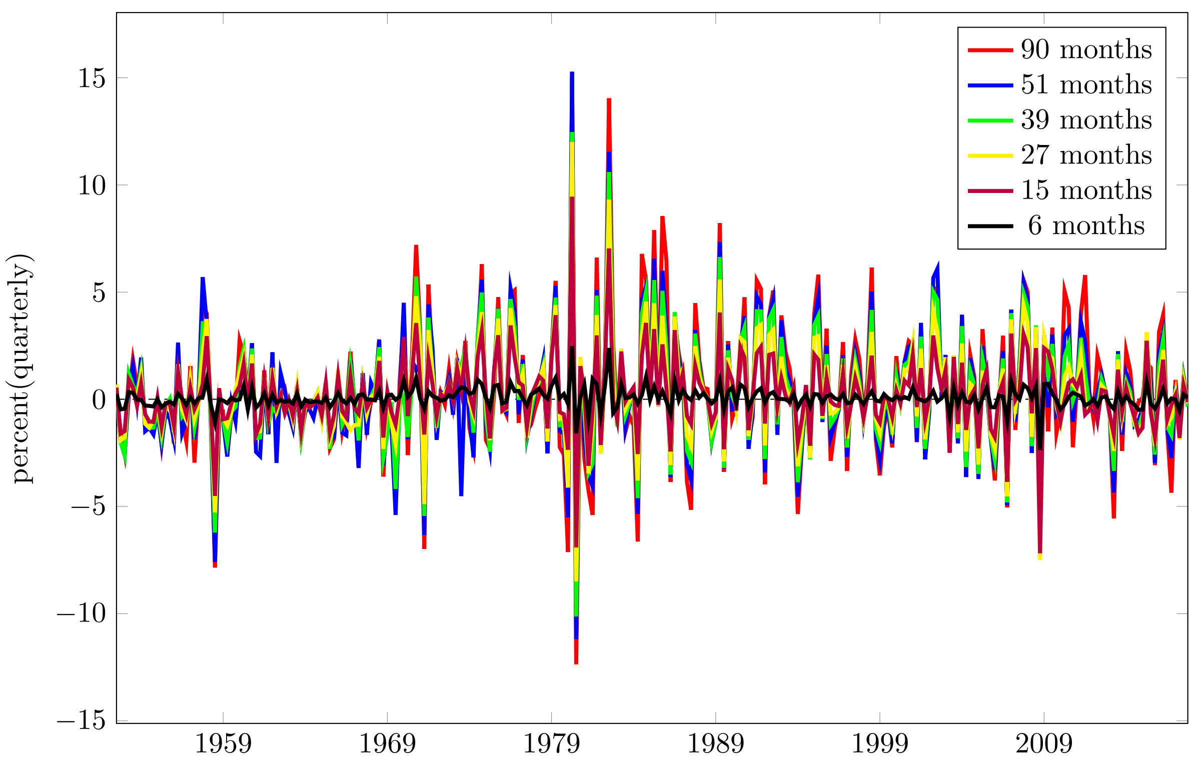


Figure 1: Time series for quarterly real holding period excess returns for a subset of maturities.

Table 2: Covariance Matrix for Real Holding Period Excess Returns and Deficits

	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	510yr	X/GDP	a_t^{rf}
hprx6m	0.21	0.39	0.59	0.71	0.81	0.87	0.89	0.91	0.92	0.95	0.97	(0.03)	0
hprx12m	0.39	0.79	1.21	1.48	1.72	1.85	1.94	2.00	2.04	2.16	2.27	0.00	0.01
hprx18m	0.59	1.21	1.88	2.31	2.69	2.92	3.06	3.17	3.24	3.45	3.61	0.04	0.01
hprx24m	0.71	1.48	2.31	2.90	3.38	3.68	3.89	4.05	4.16	4.46	4.70	0.11	0.02
hprx30m	0.81	1.72	2.69	3.38	4.06	4.40	4.68	4.90	5.04	5.46	5.80	0.15	0.02
hprx36m	0.87	1.85	2.92	3.68	4.40	4.89	5.18	5.47	5.64	6.09	6.56	0.24	0.02
hprx42m	0.89	1.94	3.06	3.89	4.68	5.18	5.63	5.94	6.15	6.65	7.23	0.30	0.03
hprx48m	0.91	2.00	3.17	4.05	4.90	5.47	5.94	6.42	6.59	7.15	7.86	0.35	0.03
hprx54m	0.92	2.04	3.24	4.16	5.04	5.64	6.15	6.59	7.01	7.52	8.28	0.40	0.03
hprx60m	0.95	2.16	3.45	4.46	5.46	6.09	6.65	7.15	7.52	8.57	9.17	0.50	0.03
hprx510yr	0.97	2.27	3.61	4.70	5.80	6.56	7.23	7.86	8.28	9.17	10.55	0.60	0.04
X/GDP	(0.03)	0.00	0.04	0.11	0.15	0.24	0.30	0.35	0.40	0.50	0.60	3.99	0.02
a_t^{rf}	0	0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.03	0.03	0.04	0.02	0.00

Notes: This table records the covariances of holding period excess returns and deficits normalized by GDP.

revenues is the sum of "federal government current tax receipts" and "federal government current receipts: contributions for government social insurance", deflated by the implicit price deflator for GDP. In table 2, we list the covariance matrix of returns, deficits (relative to GDP) and note that although the deficits are positively correlated with returns, the magnitude of the coaviance is an order of magnitude smaller than the covariances of returns with each other. Later, we will show that this feature will be key in the what shapes the optimal portfolio.

Our tax rate is measured by the sum of federal individual income tax and social security payroll tax, from the average annual marginal income-tax rates constructed by Barro and Redlick (2011). We convert annual observations from Barro and Redlick to quarterly assuming by setting the quarterly tax to the annual mean.

5.2 Objects required for the formula

We start with the constants $\left\{q, \Gamma, \zeta_T, \frac{Y_T}{B_T}\right\}$. Exploiting the stronger form stationarity, we substitute the conditional means $\mathbb{E}_T z_{T+t}$ with their sample averages. To our order of approximation, q can be estimated as an average of inverse gross return on any security. For our analysis, we use the average return on the U.S. government's debt portfolio, and get $q = 0.989$. The parameter

$$\zeta = \left(1 - \frac{\tau\gamma}{1-\tau}\right)\gamma^{-1}\left(\left(1-\overline{\tau}\right)\left[1-\tau\left(1+\gamma\right)\right]\right).$$

We set $\tau = 0.3$ which is the average tax rate from Barro and Redlick (2011), and set $\gamma = 0.5$ to recover a Frisch elasticity of 0.5. (see Chetty, 2009). We set $\frac{Y_T}{B_T} = -0.25$ and $\Gamma = 1.0025$ to reflect an debt to annual output ratio of 100%, and an annual growth rate of output to be 1%, respectively.

Liquidity premium For our formula we need to estimate the liquidity premium$\left\{a_t^j\right\}$. From equation (9), it is clear that we need to take some stand on the SDF M_t to measure liqudity premiums. In this section, we make two assumptions that simplify the construction of $\left\{a_t^j\right\}_{j,t}$.

First, we assume that only government-issued securities provide liquidity services and households' liquidity preferences are perfectly substitutable across maturities. Second, we assume that households can trade (frictionlessly) a "synthetic" risk-free bond that provides no liquidity services. These assumptions can be expressed as a special case of the utility function U_t in (5) takes the form $U_t\left(\cdot, \cdot, \sum_{i\in\mathcal{G}} q_i b_i, \cdot\right)$, with $\mathcal{G}$ being the set of securities that are issued by the government. Under these assumptions, we can use the definition (9) to express

$$1 - a_t^{rf} = \frac{q_t^{AAA,rf}}{q_t^{;rf}},$$

where $q^{AAA,rf}$ is the price of a "synthetic" risk-free bond that has no liquidity properties, and show that

$$a_t^j \simeq a_t^{rf}.$$

Thus, we just need yields on privately-issued and goverment-issed short maturities bonds $\left\{q_t^{AAA,rf}, q_t^{rf}\right\}$ to construct liquidity premiums $\left\{a_t^j\right\}_{jt}$. For $\left\{q_t^{AAA,rf}\right\}_t$ we make use of the yield curve for AAA-rated privately-issued zero coupon bonds, and for $\left\{q_t^{rf}\right\}_t$, we use the yield on the three month treasury bill. In figure (2), we plot the time-series for our constructed series for a_t^{rf}.

Orthogonal component of primary deficits For estimating Σ^X, we need to construct $\frac{X_{T+t}^{\perp}}{\overline{Y}_{T+t}}$. For any process $z_t\left(s^t\right)$, we have

$$\sigma \partial_\sigma z_{T+t} \equiv \hat{z}_{T+t} = z_{T+t} - \mathbb{E}_T z_{T+t} + O\left(\sigma\right).$$

Thus, using expression (15) to substitute for $X_{T+t}^{\perp}$, we get

$$cov_T\left(r_{T+1}^j, \frac{X_{T+t}^{\perp}}{\overline{Y}_{T+t}}\right) \simeq \sigma^2 cov_T\left(r_{T+1}^j, \frac{\partial_\sigma X_{T+t}^{\perp}}{\overline{Y}_{T+t}}\right) \simeq cov_T\left(r_{T+1}^j, \frac{\hat{X_{T+t}}}{\mathbb{E}_T Y_{T+t}} + \underbrace{\left(1 - \frac{\tau}{1-\tau} \times \gamma\right)}_{\xi} \hat{\tau}_{T+t}\right).$$

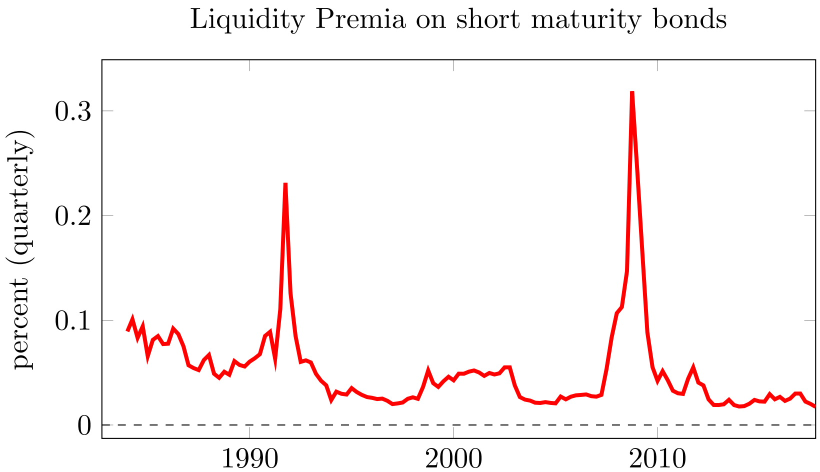


Figure 2: Time series for liquidity premia on short maturity bonds.

To construct $\frac{\hat{X_{T+t}}}{\mathbb{E}_T Y_{T+t}}$, we need to measure $\mathbb{E}_T Y_{T+t}$. Under the stationary growth assumption, $\mathbb{E}_T Y_{T+t}$ equals the trend GDP, which is recovered using $\mathbb{E}_T Y_{T+t} = e^{t\times\Gamma} Y_T$. Then, we construct $\frac{\hat{X_{T+t}}}{\mathbb{E}_T Y_{T+t}}$ as demeaned deficits relative to trend output. To measure $\hat{\tau}_{T+t}$, we detrend the tax rates constructed using Barro and Redlick (2011) data. Finally, the average revenue elasticity ξ is backed out from from the average tax rate τ, and the elasticity γ that we set to earlier to 30% and 0.5, respectively. In figure 3, we plot the time series for $\frac{X^{\perp}_{T+t}}{\mathbb{E}_T Y_{T+t}}$. Because tax rates τ_t are not that volatile, the statistical properties of $\frac{X^{\perp}_{T+t}}{\mathbb{E}_T Y_{T+t}}$ are very similar to $\frac{X_{T+t}}{\mathbb{E}_T Y_{T+t}}$.

Covariances We now use the time-series for returns, liquidity premium, and orthogonal component of deficits to construct the required covariances. A well-known concern in using inverses of covariance matrices in portfolio analysis is sampling uncertainty and how it manifests as extreme and unstable portfolio weights. For a detailed discussion, see Jagannathan and Ma (2003), DeMiguel, Garlappi, and Uppal (2007), Senneret, Malevergne, Abry, Perrin, and Jaffres (2016). These concerns apply equally to bond returns and to address them, we follow Jagannathan and Ma (2003) and Senneret, Malevergne, Abry, Perrin, and Jaffres (2016). The main idea is to exploit the fact that most of the variation in returns arises from a small set of common factors. Then we can use the estimated factor loadings to compute the covariances and their inverses.

As in Jagannathan and Ma (2003), we start a one-factor structure that implies time-state independent covariances and later extend it with a GARCH structure to have time-varying covariances. Assume that

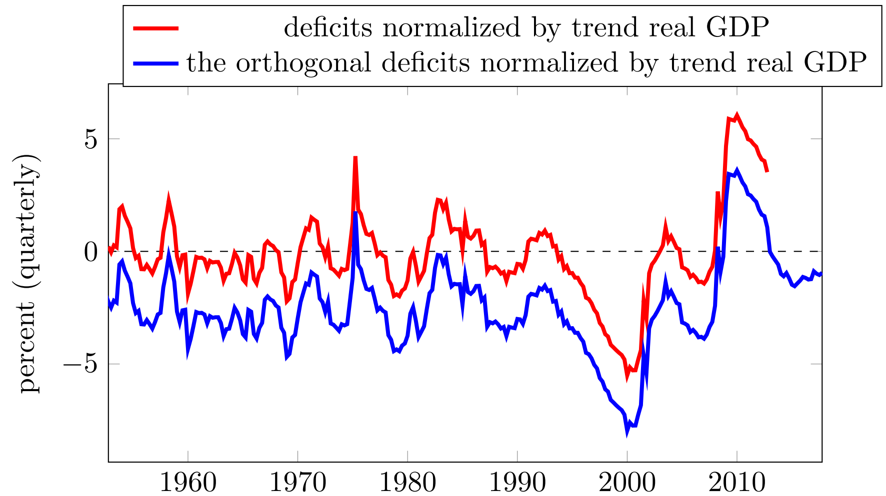


Figure 3: Time series for quarterly deficits normalized by trend real GDP using Federal Reserve Economic Data

$$r^k_{T+t} = \alpha_k + \beta_{r,k} f_{T+t} + \sigma_{r,k}\varepsilon_{k,T+t}, \tag{20}$$

$$\frac{X^{\perp}_{T+t}}{\overline{Y}_{T+t}} = \alpha_X + \rho_X \frac{X^{\perp}_{T+t-1}}{\overline{Y}_{T+t-1}} + \beta_X f_{T+t} + \sigma_X \varepsilon_{X,T+t}, \tag{21}$$

$$\ln a^{rf}_{T+t} = \alpha_a + \rho_a \ln a^{rf}_{T+t-1} + \beta_a f_{T+t} + \sigma_a \varepsilon_{a,T+t}, \tag{22}$$

$$f_{T+t} = \sigma_f \varepsilon_{f,T+t}, \tag{23}$$

where mean and variance of each of $\varepsilon_{k,T+t}$, $\varepsilon_{X,T+t}$, $\varepsilon_{a,T+t}$ and $\varepsilon_{f,T+t}$ conditional on t are 0 and 1, respectively. Using equation (20)-(23), we can show that

$$\begin{aligned} \Sigma_T[j,k] &= \beta_{r,k}\beta_{r,j}\sigma_f^2 + \mathbb{I}_{\{k=j\}}\sigma^2_{r,k} \\ \Sigma^A_T[j,k] &= \beta_a \beta_{r,j}\sigma_f^2 \frac{\left(1-\rho_a^k\right)}{1-\rho_a} \quad \Sigma^X_T[j,k] = \rho^k_X \beta_X \beta_{r,j}\sigma_f^2 \end{aligned} \tag{24}$$

Also we can show that

$$\Sigma_T^{-1} = \Delta^{-1} - \frac{\left(\Delta^{-1}\beta^r_f\right)\cdot\left(\beta^{r\intercal}_f \Delta^{-1}\right)}{\sigma_f^{-2} + \beta^{r\intercal}_f \Delta^{-1}\beta^r_f}$$

where $\Delta = \text{diag}\left\{\sigma^2_{r,j}\right\}_j$. Thus, we can back out the covariances from the estimates of the factor-model. For intermediate maturities, we interpolate the loadings $\left\{\beta_{r,j}, \sigma_{r,j}\right\}_j$, and for larger maturities outside our sample, we we set them equal to the largest maturities (i.e., 10 years) that we observe.

Table 3: FACTOR LOADINGS RETURNS

maturity	Constant	s.e.	Factor	s.e.	Adj. R2	F-stat
6	0.08	(0.02)	0.04	(0.00)	0.51	263
9	0.15	(0.03)	0.09	(0.00)	0.69	570
15	0.22	(0.04)	0.14	(0.01)	0.73	707
21	0.25	(0.05)	0.19	(0.01)	0.8	996
27	0.29	(0.05)	0.23	(0.01)	0.85	1449
33	0.32	(0.05)	0.26	(0.01)	0.89	2053
39	0.35	(0.04)	0.28	(0.01)	0.92	2984
45	0.35	(0.04)	0.3	(0.00)	0.94	4247
51	0.38	(0.04)	0.32	(0.00)	0.95	5278
57	0.31	(0.04)	0.35	(0.00)	0.95	5176
90	0.46	(0.03)	0.4	(0.00)	0.98	10147

Notes: Estimated coefficients for excess returns factor regressions (20).

We set f_t to the first principal component of returns, and estimate (20)-(23) using ordinary least squares. In table 3, we report the factor loadings on returns. Since the factor captures 90% of the variation of returns, the factor loadings are all statistically significant, and the factor loadings $\left\{\beta_{r,j}\right\}_j$ are increasing in the maturities. In table 4, we summarize the remaining estimates of equations (21) and (22). Both, the orthogonal component of deficits and the liquidity wedge on the risk-free bond are persistent. The orthogonal component loads positively on the common factor in returns, while the the liquidity wedge on the risk-free bond loads negatively on the common factor.

5.3 The Optimal Portfolio

With all the estimates in hand, we can implement the optimal portfolio formula (19). In figure 4, we plot $\{\omega_j\}$ as a function of the maturities. Overall, we obtain an exponentially declining shape. The area under the curve is 95%, which means that 5% of the debt is issued insecurity 0, or the risk-free debt. In figure 5, we investigate the sub components–portfolio that hedges interest rate risk, portfolio that hedges the primary surplus risk, and portfolio that hedges the liquidity risk. Comparing the blue and the black lines, we find that those two components offset each other. Thus, the shape of overall portfolio is largely driven the motive to hedge the interest rate risk. Overall, the fraction of total debt for hedging primary surplus risk (area above the black line in figure 5) is -17%, and an opposite position of 13% from liquidity

Table 4: FACTOR LOADINGS ORTHOGONAL COMPONENT AND RISK-FREE LIQUIDITY WEDGE

	orthogonal deficits	risk-free liquidity wedge
Constant	0.01	0.01
	(0.04)	(0.003)
Lag	0.96	0.80
	(0.02)	(0.048)
Factor	0.01	0.001
	(0.00)	(0.001)
Obs	240	131
Adj. R	0.91	0.70
F-stat	1235.04	151.4

Notes: Estimation of orthogonal deficits factor regressions (21) and liquidity wedge factor regressions (22).

risk (area under the blue line in figure 5). These magnitudes are driven by the feature that covariance of deficits (or liquidity wedge) are with returns are small relative to the covariances of returns with each other. One can gather the intuition for why they mirror each other too. Substitute the expressions (24) in formula (19) to arrive at

$$\omega = (1 - q\Gamma)\,\mathbf{w} + \frac{Y}{B}\sigma_f^2 \left[\left(\frac{\beta_X}{1 - \overline{q}\rho_X} \right) - \left(\frac{\beta_a \zeta}{1 - \rho_a} \right) \left(\frac{1}{1 - q\Gamma} - \frac{1}{1 - \rho_a \Gamma} \right) \right] \Sigma^{-1} \boldsymbol{\beta}_r.$$

The shape of the primary surplus risk component and the liquidity risk component of the optimal portfolio is inherited from the common vector $\Sigma^{-1}\boldsymbol{\beta}_r$, and the relative magnitudes depend of the term in the square bracket. Our estimates suggest that the term in the square bracket is close small because the two terms are of similar magnitude and cancel each otheer. The economic intuition for the opposite signs comes from the cyclical patterns in deficits, expected returns, and liquidity premia. After recessions, deficits are high, but also and risk premia, and liquidity premia are high for a few periods. The opposite is true after booms. Thus the signs of the conditional covariances of returns and deficits are the same as that of returns and liquidity premia.

We next compare the optimal portfolio with the U.S. debt portfolio. We use CRSP to get the amount outstanding and Macaulay duration for all federally issued (marketable) debt.[5] Then for each date, we split the outstanding debt in bins indexed by maturities (at quarterly

[5]For a few bonds where the duration is absent, we set duration equal to maturity date minus current quotation date. The CRSP database does not have outstanding amounts for bills. To address this, we supplement the CRSP data with data from Monthly Statements of Public Debt issued by the US Treasury and fill in the amount outstanding in bills.

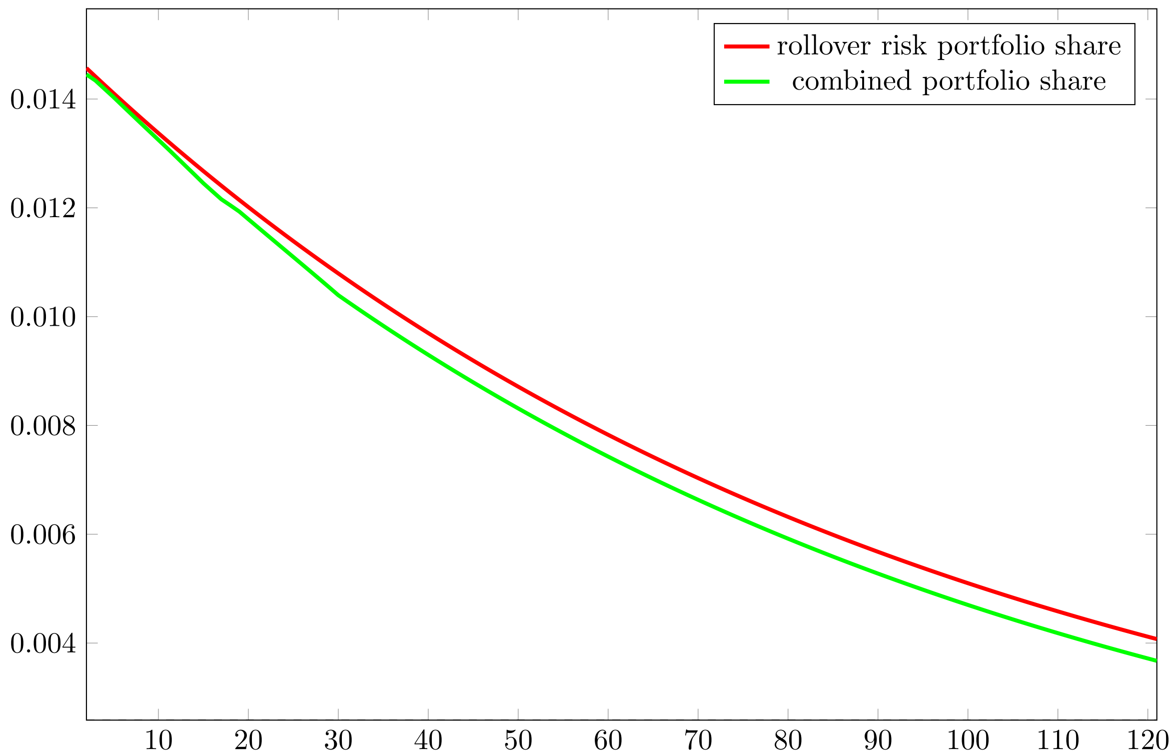


Figure 4: Portfolio shares of securities with maturities from 2 quarters to 121 quarters

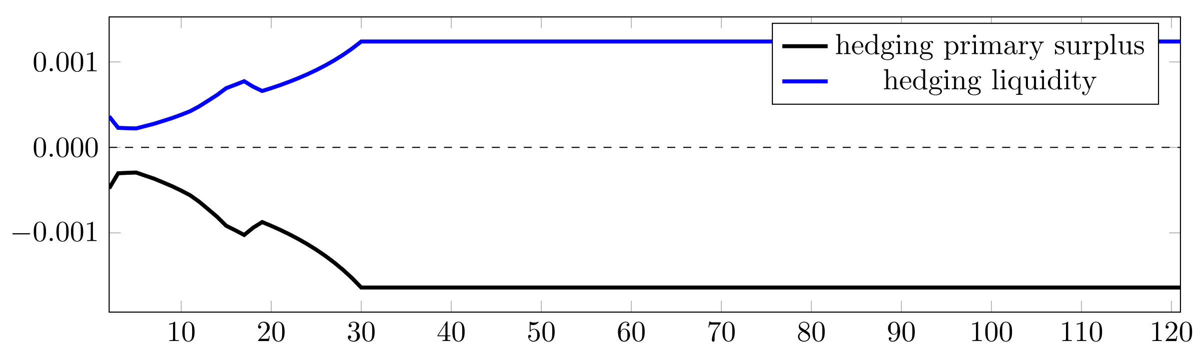


Figure 5: Sub components of optimal portfolio shares

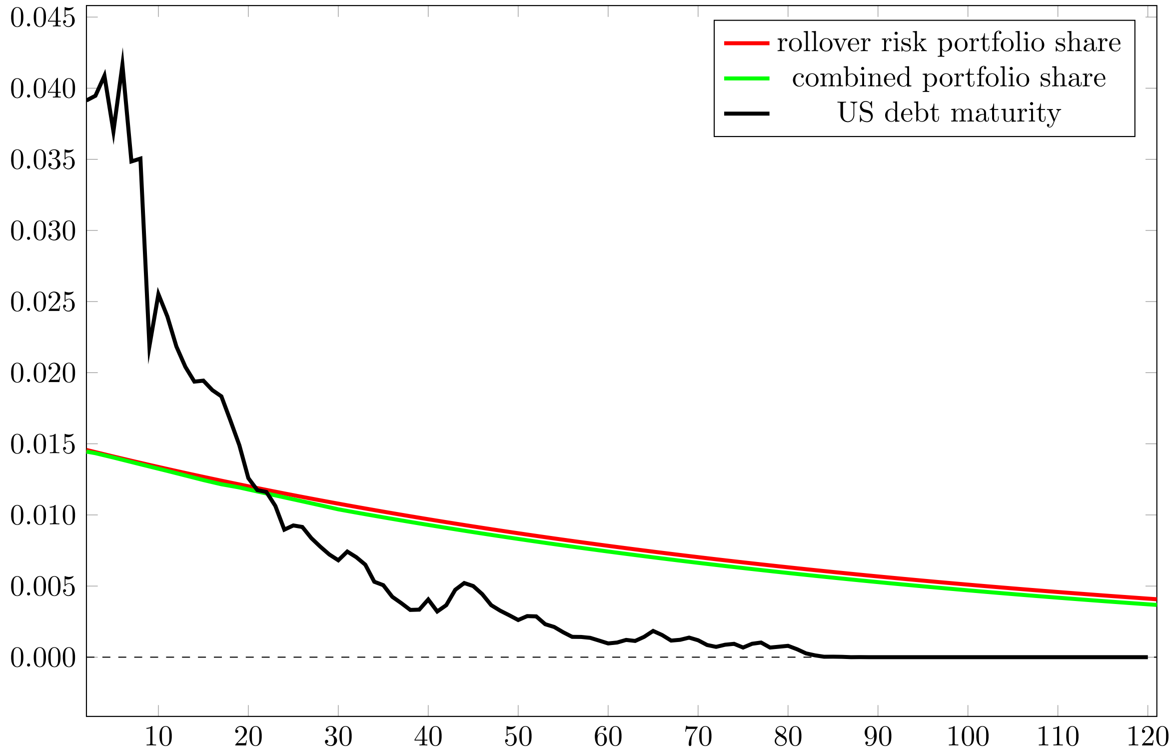


Figure 6: Portfolio shares of securities with maturities from 2 quarters to 121 quarters

intervals). In figure 6, we overlay the time-averaged US debt profile with the optimal profile. The U.S. debt profile starts above the optimal and curves cross each other at around 25 quarters. We find that the US overweights short maturities relative to the optimal. In terms of Macaulay duration, the optimal portfolio has a duration of about 13 years which is much larger than the range 5 years that we found for the U.S. debt profiles.

5.4 Time-varying covariances

The statistical model used so far implies that first and second moments are time-independent. This yielded a optimal portfolio that was stationary. Even with condition (17), our theory suggests that the portfolio that hedges interest rate risk requires no rebalancing while the other two hedging components should if $\left\{\zeta_T, \frac{Y_T}{B_T}\right\}$ or $\left\{\Sigma_T, \Sigma_T^X, \Sigma_T^A, \Sigma_T^a\right\}$ vary with time. In this section, we revisit the calculations of the optimal portfolios after allowing for time-variation in the variances and covariances of returns. We extend the factor model laid out in equations (20) – (22) so that the parameters $\left(\{\sigma_{r,k}\}_k, \sigma_X, \sigma_a, \sigma_f\right)$ are replaced by univariate GARCH processes:

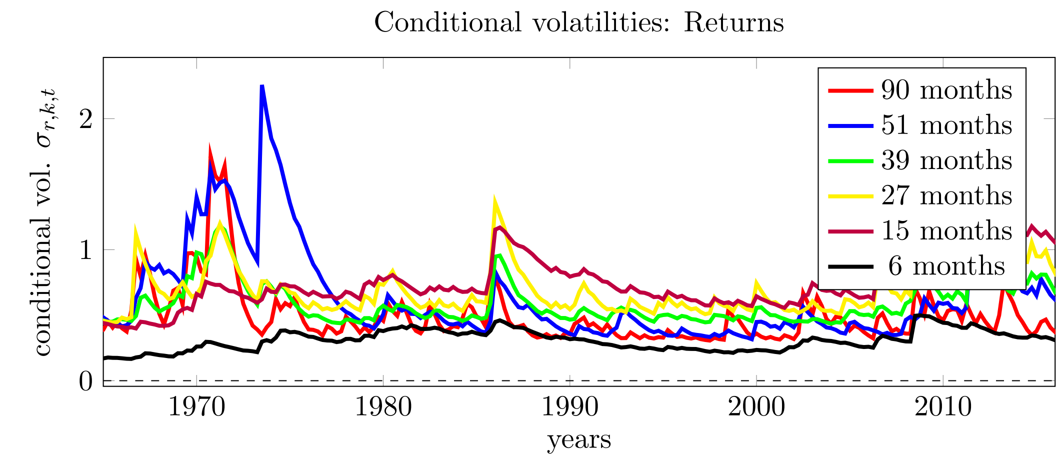
$$\sigma_{x,t}^2 = \overline{\sigma}_x + \sum_{j=1}^{P_x} \sigma_{x,p}^{AR} \varepsilon_{t-p}^2 + \sum_{j=1}^{Q_x} \sigma_{x,q}^{AR} \sigma_{t-p}^2$$

where x is a generic place holder for the returns, orthogonal deficits, risk-free liquidity wedge. We impose that all ε's are standard Gaussian and indepdnent of each other. An immediate corrolary of this structure is that

$$\Sigma_t = \sigma_{f,t+1}^2 \beta_f \beta_f^{\intercal} + \Delta_{t+1}$$

where Δ_t now is the diagonal matrix with $\{\sigma_{k,t}^2\}_k$. Similarly, we have $\Sigma_t^X [j,k] = \rho_X^k \beta_X \beta_{r,j} \sigma_{f,t+1}^2$ and $\Sigma_t^A [j,k] = \beta_a \beta_{r,j} \sigma_{f,t+1}^2 \frac{(1-\rho_a^k)}{1-\rho_a}$. We impose that ϵ_t^x is Gaussian, set $P_x = Q_x = 2$ and estimate $\left(\overline{\sigma}_x, \left\{\sigma_{x,p}^{AR}\right\}_{p \leq P_x}, \left\{\sigma_{x,q}^{MA}\right\}_{q \leq Q_x}\right)$ for all x using Maximum Likelihood.

In figure 7, we plot estimated conditional volatilities for a subset of variables. There is a clear pattern. The volatilities for returns (including the factor) and macro aggregates are high in the early 80s and the great recession of 2008-2010 and quite stable in the intervening periods. Keeping everything else the same, periods when the factor is more volatile increases the covariance of returns, the covariance of returns with output growth (or tax revenues) as well as the variance of returns. Thus, its effect on the optimal portfolio is ambiguous. In figure 8, we plot the optimal portfolio net of the portfolio that hedges the interest rate risk. Quantitatively, we find that the optimal portfolios are stable, in spite of the fact that the volatities of the returns and factors are quite different in these sub periods.



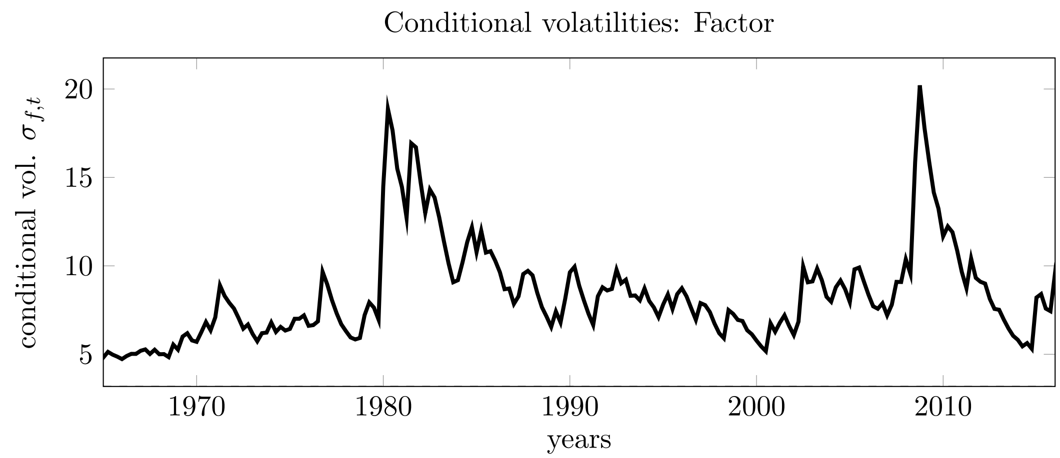


Figure 7: Conditional volatilities of returns, and the common factor using the estimated GARH model

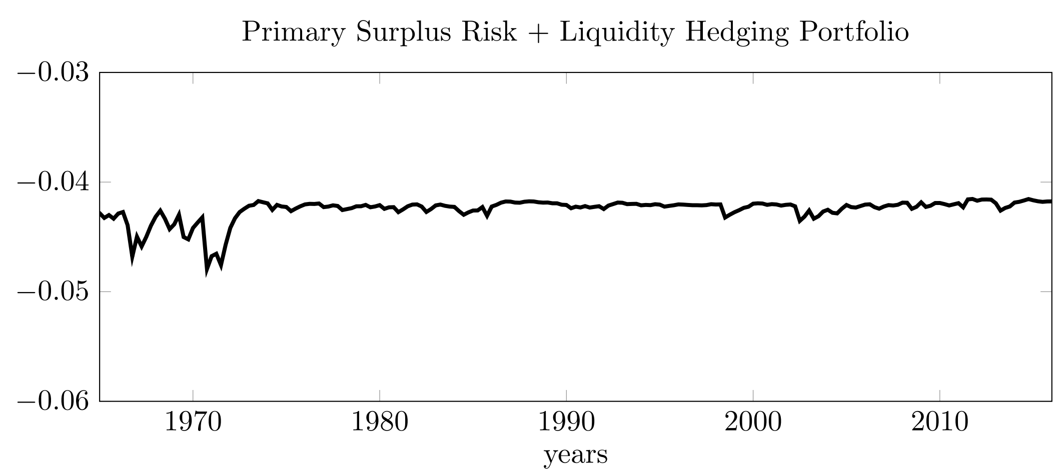


Figure 8: The sum of shares for the primary surplus and liquidity hedging portfolio

6 Optimal public portfolios and price impact

The analysis in section 4 was done under the assumption that asset prices do not respond to the adjustment's in government portfolios and subsequent tax changes. In this section we drop this assumption. In general, to study optimal public portfolios when government actions affect asset prices one needs to know how changes in composition of government portfolios and tax rates affect returns on various assets. This is ultimately an empirical question. Our theoretical framework imposes very few restrictions on how asset prices may change in response to the perturbations we considered in section 4. The literature that estimates such price impacts using credible, well-identified causal effects is still largely in its infancy, although several recent papers made important progress in answering this question.

Because of these difficulties, and a lack of a consensus on the model of asset price formation in the literature, we approach the analysis in this section as follows. We restrict attention to study the optimal portfolio of government debts, similar to the one we analyzed in section 4. We consider three different models of how bond prices are affected by government's perturbation of its portfolio. We first start with the simplest model, in which we assume that price of government debt of any given maturity is a simple downward sloping function of the outstanding quantity of that debt. This model is most simple and transparent, and will allow us to illustrate many insights very simply. This model is quite popular in both theoretical (e.g., Bigio, Nuno, and Passadore (2019)) and empirical (e.g., Hamilton and Wu (2012)) work. We then consider a model in which prices of debts of different maturities depend on the duration of government debt. This model is inspired by work of Greenwood and Vayanos (2014), who both developed a theoretical framework to account for the observed price impacts of changes in the maturity structure of public portfolios similar to those that we studied in section 4, and used U.S. data to estimate key parameters of that model. In the empirical part of this section, we show that parameters estimated by those authors map directly into the sufficient statistics required by this theory, and use their estimates to re-visit our analysis in section 5. Finally, the third model we consider is the one where we assume that there are no foreign investors, economy is closed, and households receive no direct benefits from government bonds. This model is a version of standard Ramsey models used widely to study optimal public portfolios (Lucas and Stokey (1983), Angeletos (2002), Buera and Nicolini (2004)). We use this model to illustrate two insights. First, qualitatively, the economic forces that shape optimal public portfolios are very similar to the simpler models we consider. Secondly, quantitatively, such models have strong predictions about the relationship between prices of government debts and macroeconomic variables, and how such prices respond to changes in government portfolios.

Such predictions are difficult to reconcile with the U.S. data.

6.1 Simple price effects

In this section, we assume that government faces a simple downward demand curve for each maturity of debt that it issues. That is, we assume that the relationship between prices and quantities for each security are given by

$$\ln q_t^i = \lambda_{0,t}^i + \lambda_t^i B_t^i. \tag{25}$$

It is natural to assume is that $\lambda_t^i \geq 0$, so prices decline in the amount of outstanding debt (recall, that under our sign convention, debt is a negative value of B_t^i).

Consider the same perturbation introduced in the beginning of section 4: the government issues $\frac{\epsilon}{q_T^j}$ units of any debt j in period T than it then repays in period $T+1$, with adjusting taxes in all periods to satisfy its budget constraint. Let $\phi_t^i \equiv -b_t^i/B_t^i$ be the fraction of government debt i held by households. The welfare effect from this transaction be can written as

$$\begin{aligned} \frac{\partial_\epsilon V_0}{\beta^T \Pr\left(s^T\right) M_T\left(s^T\right)} = &\left\{\left(\frac{1}{\xi_T} - 1\right) - \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} R_{T+1}^j \left(\frac{1}{\xi_{T+1}} - 1\right)\right\} + a_T^j \left(1 + \lambda_T^j \phi_T^j B_T^j\right) \\ &+ \lambda_T^j \left[\left(\frac{1}{\xi_T} - \phi_T^j\right) B_T^j - \left(\frac{1}{\xi_T} - \phi_{T-1}^j\right) B_{T-1}^j\right]. \end{aligned} \tag{26}$$

It is instructive to compare this expression to equation (10). The term in the curly brackets is identical in the two cases and captures the smoothing of deadweight losses from taxation. The second force is the liquidity term a_T^j, but it is now adjusted by $\left(1 - \lambda_T^j b_T^j\right)$. Recall that when a government sells one dollar worth of government bond j, it generates a_T^j units of welfare due to the liquidity premium. However, selling bond also lowers its price, and therefore the market value of holding of that security. This offsets some of the benefits from the additional liquidity previous. That, all things being equal, welfare is higher if liquidity is provided via price-insensitive bonds, that have low value of λ_T^j.

Finally, the term in the second line of (27) captures welfare impact of additional revenues raised or lost due to price changes. To understand this term, observe that price change has a net effect $\lambda_T^j \left(B_T^j - B_{T-1}^j\right)$ on government budget constraint. Selling $\frac{\epsilon}{q_T^j}$ units of security lowers its price by λ_T^j. This has no direct resource effect on government budget constraint if, besides engaging in this transaction, government did not trade this security, so that $B_T^j - B_{T-1}^j = 0$ in the baseline equilibrium. If in the baseline equilibrium, the government trades security j in period T, then lower prices lead to a shortfall of $\lambda_T^j \left(B_T^j - B_{T-1}^j\right)$ dollars of resources if $B_T^j - B_{T-1}^j < 0$ (i.e. government sold security j), and a surplus if it is bought it in period T.

The shadow cost of extra resources is $\beta^T M_T/\xi_T$. Similar arguments imply that the marginal effect of the price impact for households is $\beta^T M_T \lambda_T^j \left(b_T^j - b_{T-1}^j\right)$. By combining these two effects and re-arranging, we get the second line in equation (27).

Another way to understand the price impact is to observe that we can write

$$\left(\frac{1}{\xi_T} - \phi_T^j\right) B_T^j = \left[\phi_T^j \left(\frac{1}{\xi_T} - 1\right) + \left(1 - \phi_T^j\right) \frac{1}{\xi_T}\right] B_T^j.$$

Fraction ϕ_T^j of government debt is held by domestic households. For such bonds, a dollar gain from the price impact for the government is a dollar loss for the households. The net welfare effect is not zero, however, since this extra dollar of revenues allows the government to decrease tax rates and lower distortions. Therefore, the welfare effect from the price impact on domestically held bond is proportional to the deadweight loss from taxes, $\left(\frac{1}{\xi_T} - 1\right)$. Fraction $\left(1 - \phi_T^j\right)$ of government debt is held by foreign investors. Since the government does not value income in the hand of the foreign investors, the welfare effect from the price impact on bonds held by foreigners is proportional to $\frac{1}{\xi_T}$.

Similarly to the analysis in section 4, to consider optimal portfolio, we combine the optimality condition (27) for bonds j and rf, and set the net welfare impact to zero. This gives us the optimality condition

$$\begin{aligned}
\mathbb{E}_T \frac{\beta M_{T+1}}{M_T} \frac{r_{T+1}^j}{\xi_{T+1}} &= \left\{\lambda_T^j \left(\frac{1}{\xi_T} - \phi_T^j + a_T^j \phi_T^j\right) B_T^j - \lambda_T^j \left(\frac{1}{\xi_T} - \phi_{T-1}^j\right) B_{T-1}^j\right\} \\
&\quad - \left\{\lambda_T^{rf} \left(\frac{1}{\xi_T} - \phi_T^{rf} + a_T^{rf} \phi_T^{rf}\right) B_T^{rf} - \lambda_T^{rf} \left(\frac{1}{\xi_T} - \phi_{T-1}^{rf}\right) B_{T-1}^{rf}\right\}. \quad (27)
\end{aligned}$$

The underlying transaction we consider is very similar in flavor to Quantitative Easing (QE) that central banks of several countries conducted in the aftermath of the 2007 financial crises, when they issued short debt to purchase debts of longer maturities. We then approximate this optimality condition to obtain a tractable extension to our optimal portfolio formula.

Before presenting the expansion, it is useful to point out that we need to take a stance on how parameter λ_t^i depends on σ. In what follows, we assume that our transaction generates a price impact of the order of $O\left(\sigma^2\right)$, so that $\lambda_t^i = O\left(\sigma^2\right)$. We do it for two reasons. If price impact is of order lower than σ^2, then the composition of the optimal public portfolio is determined essentially exclusively by price impacts.[6] More importantly, many commonly used models of asset price determination imply the price impact from QE-style asset swap should be zero to the zeroth and first-order approximations. For example, this is true both in Greenwood and Vayanos (2014) that we study in section 6.2 and in closed economy that we analyze in

[6]See Bigio, Nuno, and Passadore (2019) who analyses a model that closely resembles this case.

section 6.3. To the zeroth order, there is no risk, and to the first order all risk is price by risk-neutral agents, so in both cases the transaction we consider involves swapping to assets with identical expected returns, and all economic agents are indifferent about such swaps.

Applying our approximation, the analogue of equation (11) becomes

$$
\begin{aligned}
cov_T\left(\ln \xi_{T+1}, r^j_{T+1}\right) \simeq & \left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}}\right)^{-1}\left[a^{rf}_T - a^j_T\right] \\
& - \left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}}\right)^{-1}\left(\overline{\frac{\xi_{T+1}}{2}}\right)\left\{\lambda^j_T\left(\frac{1}{\overline{\xi}_T} - \overline{\phi}^j_T + \overline{a}^j_T\overline{\phi}^j_T\right)\overline{B}^j_T - \lambda^j_T\left(\frac{1}{\overline{\xi}_T} - \overline{\phi}^j_{T-1}\right)\overline{B}^j_{T-1}\right\} \\
& + \left(\overline{\frac{\xi_{T+1}}{2}}\right)\left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}}\right)^{-1}\left\{\lambda^{rf}_T\left(\frac{1}{\overline{\xi}_T} - \overline{\phi}^{rf}_T + \overline{a}^{rf}_T\overline{\phi}^{rf}_T\right)\overline{B}^{rf}_T - \lambda^{rf}_T\left(\frac{1}{\overline{\xi}_T} - \overline{\phi}^{rf}_{T-1}\right)\overline{B}^{rf}_{T-1}\right\}
\end{aligned}
\tag{28}
$$

A generalized transaction, when the government rolls over excess returns for additional k periods rather than returning them in period $T+1$, has very similar optimality condition, except there is an additional term $cov_T\left(\mathrm{A}^k_{T+1}, r^j_{T+1}\right)$ on the left hand side of (28). A convenient simplification is that all price impacts from portfolio adjustments between periods $T+1$ and $T+1+k$ disappear from the second order approximation: those portfolio adjustments are proportional to $r^j_{T+1}\lambda^{rf}_{T+1+t}$ and are of order $O\left(\sigma^3\right)$.

It is straightforward to apply the same steps as in the proof of theorem 1 to obtain optimal portfolio with price impacts. We provide this analysis in the appendix. Here we focus on additional implications that emerge due to price effects. To illustrate them transparently, we make several assumptions. We make the same stationarity assumptions (17) as we did in the proof of corollary 2. In addition, we assume that households hold a fixed proportion of government debt of each maturity, $-b^i_t/B^i_t \approx \phi$ for all i, and $t \in \{T, T-1\}$, and liquidity premium is the same for all bonds, $a^i_T \simeq a$ for all i. None of these assumptions have material impact on the points that we want to raise. Finally, we assume that $\lambda^{rf}_T = 0$, so that the one period bond is the least price sensitive among all government bonds. This is consistent with the empirical evidence (see Krishnamurthy and Vissing-Jorgensen (2011) and also D'Amico and King (2013)), and making this assumption will allow to be more concrete about the direction of impact of the price effects on the optimal portfolio.

As in section 4.1, we assume that government portfolio consists of pure discount bonds of all maturities. Let $B^{(T+t)}_T$ denote the quantity of bond that maturities in period $T+t$ that is held at period T. Let $\boldsymbol{\omega}_T$, $\hat{\boldsymbol{\omega}}_{T-1}$ be a vector with elements $\boldsymbol{\omega}_T[t] = \frac{q^{(T+1+t)}_T B^{(T+1+t)}_T}{B_T}$ and $\hat{\boldsymbol{\omega}}_{T-1}[t] = \frac{q^{(T+1+t)}_T B^{(T+1+t)}_{T-1}}{B_T}$. That is, $\boldsymbol{\omega}_T[t]$ is the portfolio share of the bond that matures in period $T+1+t$ in period T portfolio, while $\hat{\boldsymbol{\omega}}_{T-1}[t]$ is a counterfactual portfolio share of the same bond if the government kept its quantity the same as in period $T-1$.Thus, $\boldsymbol{\omega}_T[t]-\hat{\boldsymbol{\omega}}_{T-1}[t]$ the market value of rebalancing holdings of bond that matures in period $T+1+t$ adjusted by the market value of debt in period T. Finally, let Λ_T be a diagonal matrix with elements $\Lambda_T[t,t]=\frac{Y_T\lambda_T^{(T+1+t)}}{q_T^{(T+1+t)}}$. We have

Proposition 3. *Under the assumptions of this section, the optimal portfolio of public debts satisfies*

$$\begin{aligned}\boldsymbol{\omega}_T \simeq (1-q\Gamma)\,\mathbf{w} &+ \left(\frac{Y_T}{qB_T}\right)\Sigma_T^{-1}\left[\Sigma_T^X+\zeta_T\left(\frac{q^{-1}}{1-a_T^{rf}}\Sigma_T^a-q\Gamma\Sigma_T^A\right)\right]\mathbf{w} \\ &- \chi_T\Sigma_T^{-1}\Lambda_T\left[\left(\frac{1}{\xi_T}-\phi\right)(\boldsymbol{\omega}_T-\hat{\boldsymbol{\omega}}_{T-1})\right]. \end{aligned} \tag{29}$$

where expression for $\chi_T>0$ is given in the appendix.

The first line on this expression is identical to (19) and shows the same three hedging motives – interest rates, deficits, and liquidity risk – as in the previous section. The second line shows the additional consideration: the cost of portfolio rebalancing, that is proportional to $(\boldsymbol{\omega}_T-\hat{\boldsymbol{\omega}}_{T-1})$. An alternative way to write equation (19) is to use the definition of $\mathbf{w}_T^*$.

Corollary 4. *The optimal portfolio satisfies*

$$\boldsymbol{\omega}_T=\mathbf{w}_T^*-\underbrace{\chi_T\Sigma_T^{-1}\Lambda_T\left(\frac{1}{\bar{\xi}_T}-\phi\right)}_{D_T}(\boldsymbol{\omega}_T-\hat{\boldsymbol{\omega}}_{T-1})\,.$$

This corollary provides a way to think about the law of motion of the optimal portfolio. Each period, there is the optimal portfolio target $\mathbf{w}_T^*$. This is where ideally the government would like the portfolio to be, if there were no cost of rebalancing. In the stationary economy, this portfolio is same as the one that hedges the three risks to the government. Price effects also have implications for how quickly the government goes to that target. Large price effects imply higher D_T and hence slower speed of convergence.

In the previous section, we showed that the expression in square brackets in equation (29) is small empirically, and without price effects the optimal portfolio is approximately given by $(1-q\Gamma)\,\mathbf{w}$. One property of this portfolio is that in the stationary economy it requires no rebalancing: if $\boldsymbol{\omega}_{T-1}=(1-q\Gamma)\,\mathbf{w}$, then $\hat{\boldsymbol{\omega}}_{T-1}=(1-q\Gamma)\,\mathbf{w}$ and therefore $\boldsymbol{\omega}_T=(1-q\Gamma)\,\mathbf{w}$. Minimizing rebalancing is exactly the feature the minimizes rollover needs and therefore interest risk. This property is summarized in the next lemma

Corollary 5. *If the expression in the square brackets in equation (29) is equal to* $\mathbf{0}$ *and economy is stationary, then*

$$\boldsymbol{\omega}_T = (1 - q\Gamma)\,\mathbf{w}.$$

A more general version of the formula that corresponds to (16) can be obtained if we dispense with stationarity. We use this generalization to explain how the optimal portfolio with price effects depends on liquidity premia. We can show that the

$$\begin{aligned} q_T^{rf}\frac{B_T}{Y_T}\Sigma_T\boldsymbol{\omega_T} \simeq & \left[\Sigma_T^Q\Pi_T^Q + \Sigma_T^X + \left(\Sigma_T^a\Pi_T^a - \Sigma_T^A\Pi_T^A\right)\right]\mathbf{w}_T \\ & - \left(\frac{q_T^{rf}B_T}{\bar{Y}_T}\right)\chi_T\Lambda_T\left[\left(\frac{1}{\bar{\xi}_T} - \phi\right)(\boldsymbol{\omega_T} - \hat{\boldsymbol{\omega}}_{\boldsymbol{T-1}})\right] \\ & - \left(\frac{q_T^{rf}B_T}{\bar{Y}_T}\right)\chi_T\Lambda_T a_T\phi\boldsymbol{\omega_T} \end{aligned}$$

The first line is same as the case with no price effects (16). The second line is the same as (29) and captures the cost of adjustment. The new term in the third line that depends on the liquidity premium. The new term implies that the portfolio weights are adjusted downward when $\overline{a}_T > 0$. The reason for the downward adjustment can be understood from our discussion following equation (27): it is better to provide liquidity with assets that are less price sensitive. Since we assumed that one period bond is least price sensitive, the government issues more of this very short bond and less of bonds of longer maturities.

6.2 Greenwood-Vayanos price effects

In this section we consider the implications of price effects implied by the work of Greenwood and Vayanos (2014). Those authors developed a framework to account for observed price responses to changes in the composition of government debt and estimated their key parameters using U.S. data. In their model, government debt is prices by “arbitrageurs” that are equivalent to our foreign investors. Those arbitrageurs are the marginal investors who purchase debts of different maturities to maximize their mean-variance utility. “Other investors” can hold some of government debt but trading frictions prevent them from pricing debt on the margin. Those are equivalent to our households with trading frictions causing non-zero liquidity wedge.

Greenwood and Vayanos show that in their model price of each debt is a function of the overall duration of government portfolio, and in the empirical work they consider price functions of the form

$$\ln q_T^{(T+t)} = \lambda_{0,T} + \lambda_T^{(t)}\left(\sum_{k=1}^{\infty} kB_T^{(T+k)}\right) \tag{30}$$

where the expression in the brackets in the (negative of) duration of outstanding portfolio of government bonds.

It is easy to verify that in their model $\lambda_T^{(t)}$ satisfy the restrictions we impose in section 6.1: $\lambda_T^{(t)}$ are of the order $O\left(\sigma^2\right)$ and the shortest maturity is price insensitive, corresponding to $\lambda_T^{(1)} = 0$ in our discrete time model.

It is easy to see that our analysis from section 6.1 extends with minimal changes to these settings. Consider the portfolio swap we discussed above, whereas the government buys ϵ units of bond of duration $t+1$ and decreases holding of a one period bond by $-q_T^{(1+t)}/q_T^{(1)}\epsilon$. It is easy to verify that this increases duration, by

$$\partial_\epsilon D_T = \left[\frac{1+t}{q_T^{(T+1+t)}} - \frac{1}{q_T^{(T+1)}}\right] q_T^{(1+t)}.$$

and hence prices of each security change by $\partial_\epsilon q_T^{(T+t)} = \lambda_T^{(t)} q_T^{(T+t)} \left[\frac{1+t}{q_T^{(1+t)}} - \frac{1}{q_T^{(1)}}\right] q_T^{(1+t)}$. Using the same steps as in the previous section, we establish

Proposition 6. *Suppose prices are given by (30). Then formula (29) characterizing optimal portfolio holds, except matrix Λ_T has elements*

$$\Lambda_T[i,j] = \left(\frac{1+i}{q_T^{(T+1+i)}} - \frac{1}{q_T^{(T+1)}}\right)\frac{\lambda_T^{(j)} Y_T}{q_T^{(T+1+j)}}. \tag{31}$$

Thus, the only thing that changes with price prices of the form (30) is the quantitative properties about the target portfolio and the speed of reversion to it, that depends on matrix Λ_T. Qualitatively, all the results of the previous section remain unchanged.

We finally use estimates from Greenwood and Vayanos (2014) to measure $\Lambda_T[i,j]$ from the estimates of a set of regressions of yields on duration of public debt that are reported in their paper. More specifically, they regress

$$\ln y_t^n = a^n + b^n D_t - c^n \ln y_t^1 + \text{noise}$$

where is the duration of public debt. Using the equation (30), we get that $\lambda_T^{(n)} = -n \times b^n$. While point estimates of b^n vary across maturities, they are not that different statistically. We therefore, set all b^ns equal to the 0.003, which is the mean across all maturities that they report. We also need to take a stand on how the debt is split between domestic and foreign holders. In Greenwood and Vayanos (2014) setup, the marginal debt issuances are to foreigners. For now, we capture this segmentation by assuming that $b_{i,t} = 0$.

Putting everything together, the mean that the target portfolio in equation (29) simplifies to

$$\left(I + x\Sigma^{-1}\Lambda\right)^{-1}\left[\omega^{no,pe}\right] = \boldsymbol{\omega}^{target}$$

where the $\omega^{np,pe}$ is the stationary portfolio when price impact is turned off; the constant x is given by

$$x = \frac{(1 - \tau - \gamma\tau)^2}{\gamma} \times \frac{\Gamma}{1 - q\Gamma}.$$

and the i, j element of matrix Λ are described in by equation (31). We find that the two are very similar to each other. This follows from corollory 5 and our earlier finding that the portfolio without price effects mirrors the portfolio that hedges the interest rate risk.

6.3 Price effects in a closed economy

To be written

7 Conclusion

We study the optimal composition of a government's portfolio in a large class of macro-finance models. We derive a formula for the optimal portfolio and show that it can be expressed in terms of estimable "sufficient statistics". We use U.S. data to calculate the key moments required by our theory and show that they imply that the optimal portfolio is approximately geometrically declining in bonds of different maturities and requires little rebalancing in response to aggregate shocks. Our optimal portfolio differs from portfolios prescribed by existing models often used in the business cycle literature and also from those adopted by the U.S. Treasury. The key normative differences are driven by counterfactual asset pricing implications of the standard models. A natural extension to our exercise is apply our methods in settings in which the government lacks commitment. Such models are extensively in the international finance literature to study sovereign debt. We leave this for future work.

References

Ai, H., and R. Bansal (2018): "Risk Preferences and the Macroeconomic Announcement Premium," *Econometrica*, 86(4), 1383–1430.

Albuquerque, R., M. Eichenbaum, V. Luo, and S. Rebelo (2016): "Valuation Risk and Asset Pricing," *Journal of Finance*, 71(6), 2861–2904.

Angeletos, G.-M. (2002): "Fiscal Policy with Noncontingent Debt and the Optimal Maturity Structure," *Quarterly Journal of Economics*, 117(3), 1105–1131.

Barro, R. J., and C. J. Redlick (2011): "Macroeconomic Effects from Government Purchases and Taxes," *Quarterly Journal of Economics*, 126(1), 51–102.

Bhandari, A., D. Evans, M. Golosov, and T. Sargent (2021): "Inequality, Business Cycles, and Monetary-Fiscal Policy," Working paper.

Bigio, S., G. Nuno, and J. Passadore (2019): "A Framework for Debt-Maturity Management," Working Paper 25808, National Bureau of Economic Research.

Buera, F., and J. P. Nicolini (2004): "Optimal Maturity of Government Debt without State Contingent Bonds," *Journal of Monetary Economics*, 51(3), 531–554.

Chetty, R. (2009): "Sufficient Statistics for Welfare Analysis: a Bridge Between Structural and Reduced-Form Methods," *Annual Review of Economics*, 1(1), 451–488.

D'Amico, S., and T. B. King (2013): "Flow and stock effects of large-scale treasury purchases: Evidence on the importance of local supply," *Journal of Financial Economics*, 108(2), 425–448.

Debortoli, D., R. Nunes, and P. Yared (2017): "Optimal Time-Consistent Government Debt Maturity," *Quarterly Journal of Economics*, 132(1), 55–102.

DeMiguel, V., L. Garlappi, and R. Uppal (2007): "Optimal versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy?," *Review of Financial Studies*, 22(5), 1915–1953.

Devereux, M. B., and A. Sutherland (2011): "Country Portfolios in Open Economy Macro-Models," *Journal of the European Economic Association*, 9(2), 337–369.

Epstein, L. G., and M. Schneider (2003): "Recursive Multiple-Priors," *Journal of Economic Theory*, 113(1), 1–31.

Epstein, L. G., and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: a Theoretical Framework," *Econometrica*, 57(4), 937–969.

Ergin, H., and F. Gul (2009): "A Theory of Subjective Compound Lotteries," *Journal of Economic Theory*, 144(3), 899–929.

Faraglia, E., A. Marcet, R. Oikonomou, and A. Scott (2018): "Government Debt Management: the Long and the Short of It," *Review of Economic Studies*, pp. 2554–2604.

Farhi, E. (2010): "Capital Taxation and Ownership when Markets Are Incomplete," *Journal of Political Economy*, 118(5), 908–948.

Gilboa, I., and D. Schmeidler (1989): "Maxmin Expected Utility With Non-Unique Prior," *Journal of Mathematical Economics*, 18(2), 141–153.

Greenwood, R., and D. Vayanos (2014): "Bond Supply and Excess Bond Returns," *Review of Financial Studies*, 27(3), 663–713.

Gul, F. (1991): "A Theory of Disappointment Aversion," *Econometrica*, 59(3), 667–686.

Hamilton, J. D., and J. C. Wu (2012): "The Effectiveness of Alternative Monetary Policy Tools in a Zero Lower Bound Environment," *Journal of Money, Credit and Banking*, 44(s1), 3–46.

Hansen, L. P., and T. J. Sargent (2008): *Robustness*. Princeton University Press.

Hayashi, T., and J. Miao (2011): "Intertemporal Substitution and Recursive Smooth Ambiguity Preferences," *Theoretical Economics*, 6(3).

Jagannathan, R., and T. Ma (2003): "Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps," *Journal of Finance*, 58(4), 1651–1683.

Klibanoff, P., M. Marinacci, and S. Mukerji (2005): "A Smooth Model of Decision Making under Ambiguity," *Econometrica*, 73(6), 1849–1892.

——— (2009): "Recursive smooth ambiguity preferences," *Journal of Economic Theory*, 144(3), 930–976.

Kreps, D. M., and E. L. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46(1), 185–200.

Krishnamurthy, A., and A. Vissing-Jorgensen (2011): "The Effects of Quantitative Easing on Interest Rates: Channels and Implications for Policy," *Brookings Papers on Economic Activity*, pp. 215 – 265.

——— (2012): "The Aggregate Demand for Treasury Debt," *Journal of Political Economy*, 120(2), 233–267.

Kyle, A. S. (1985): "Continuous Auctions and Insider Trading," *Econometrica*, 53(6), 1315–1335.

Lucas, R. E., and N. L. Stokey (1983): "Optimal Fiscal and Monetary Policy in an Economy without Capital," *Journal of Monetary Economics*, 12(1), 55–93.

Lustig, H., C. Sleet, and S. Yeltekin (2008): "Fiscal Hedging with Nominal Assets," *Journal of Monetary Economics*, 55(4), 710 – 727.

Maccheroni, F., M. Marinacci, and A. Rustichini (2006a): "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, 74(6), 1447–1498.

——— (2006b): "Dynamic Variational Preferences," *Journal of Economic Theory*, 128(1), 4–44.

Samuelson, P. A. (1970): "the Fundamental Approximation Theorem of Portfolio Analysis in terms of Means, Variances and Higher Moments," *Review of Economic Studies*, 37(4), 537–542.

Schmitt-Grohe, S., and M. Uribe (2004): "Optimal Fiscal and Monetary Policy under Sticky Prices," *Journal of Economic Theory*, 114(2), 198–230.

Senneret, M., Y. Malevergne, P. Abry, G. Perrin, and L. Jaffres (2016): "Covariance Versus Precision Matrix Estimation for Efficient Asset Allocation," *IEEE Journal of Selected Topics in Signal Processing*, 10(6), 982–993.

Strzalecki, T. (2011): "Axiomatic Foundations of Multiplier Preferences," *Econometrica*, 79(1), 47–73.

Viceira, L. M. (2001): "Optimal Portfolio Choice for Long Horizon Investors with Nontradable Labor Income," *Journal of Finance*, 56(2), 433–470.

8 Appendix

8.1 Proofs for section 4

We first prove the following preliminary result.

Lemma 7. $\xi_t \in (0, 1+\gamma_t)$, $M_t > 0$.

Proof. Function $\frac{\tau}{1-\tau}$ is strictly increasing on $(-\infty, 1)$ interval and its range is $(-1, \infty)$. Tax revenues are $\tau (1-\tau)^{\gamma_t} \theta_t$. It is easy to verify that tax revenues are maximized if $\frac{\tau_t}{1-\tau_t} = \frac{1}{\gamma_t}$, so the definition of regular equilibrium implies that $\frac{\tau_t}{1-\tau_t}$ is bounded away from $\frac{1}{\gamma_t}$. Therefore, $\xi_t = 1 - \gamma_t \frac{\tau_t}{1-\tau_t} \in (0, 1+\gamma_t)$.

The first order condition for household is

$$\frac{\partial V_0}{\partial V_1 (s^1)} \times ... \times \frac{\partial V_{t-1} \left(s^{t-1}\right)}{\partial V_t (s^t)} U_{c,t} \left(s^t\right) = \Pr \left(s^t\right) M_t \left(s^t\right).$$

Since $\mathbb{W}_t$ is strictly increasing, and U_t is strictly increasing in consumption, the left hand side of this equation is strictly positive, and therefore $M_t \left(s^t\right) > 0$. $\square$

We now consider properties of approximations. Thought this section we refer to approximations where uncertainty is shut down after state s^T. All our approximations work through the following line of reasoning. Consider any equilibrium condition of the form $F(\mathbf{x}_T) = 0$, where $\mathbf{x}_T$ is a vector of equilibrium variables and F is some function. When we consider sequences of economies parameterized by σ, this equation should hold for all σ, so it can be written as $\mathbb{E}_T F(\mathbf{x}_{T+1}(\sigma)) = 0$. When we expand it with respect to σ, we get

$$F(\bar{\mathbf{x}}_T) + \sigma F_x (\bar{\mathbf{x}}_T) \partial_\sigma \bar{\mathbf{x}}_T + \frac{\sigma^2}{2} (\partial_{\sigma\sigma} \bar{\mathbf{x}}_T)^{\mathsf{T}} F_{xx} (\bar{\mathbf{x}}_T) \partial_{\sigma\sigma} \bar{\mathbf{x}}_T + O\left(\sigma^3\right) = 0,$$

where $F_x(\bar{\mathbf{x}}_T)$ and $F_{xx}(\bar{\mathbf{x}}_T)$ are Jacobian and Hessian of F evaluated at $\bar{\mathbf{x}}_T$. Since this equation should hold for all σ, we must have

$$F(\bar{\mathbf{x}}_T) = F_x(\bar{\mathbf{x}}_T) \times \partial_\sigma \bar{\mathbf{x}}_T = \mathbb{E}_T (\partial_{\sigma\sigma} \bar{\mathbf{x}}_T)^{\mathsf{T}} F_{xx}(\bar{\mathbf{x}}_T) \partial_{\sigma\sigma} \bar{\mathbf{x}}_T = 0.$$

We use these condition (and their generalization for period $T+t$ variables) to derive properties of approximations $\bar{\mathbf{x}}_T$, $\partial_\sigma \bar{\mathbf{x}}_T$, $\partial_{\sigma\sigma} \bar{\mathbf{x}}_T$. Once we obtain these, we have the approximation of our equilibrium objects

$$\mathbf{x}_T = \mathbf{x}_T(1) \simeq \bar{\mathbf{x}}_T + \partial_\sigma \bar{\mathbf{x}}_T + \frac{1}{2} \partial_{\sigma\sigma} \bar{\mathbf{x}}_T.$$

Lemma 8. *(i).* $\bar{r}^j_{T+1} = \mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$, $\bar{a}^{rf}_T = \bar{a}^j_T, \partial_\sigma a^{rf}_T = \partial_\sigma a^j_T$ *for all* j.

(ii). Optimality conditions (11) and its generalization for k period roll over of excess returns imply that

$$\mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r^j_{T+1} = \frac{\bar{R}^{rf}_{T+1}}{1-\bar{a}^{rf}_T} \frac{\partial_{\sigma\sigma} a^{rf}_T - \partial_{\sigma\sigma} a^j_T}{2},$$

$$\mathbb{E}_T \partial_\sigma \ln \xi_{T+1+k} \partial_\sigma r^j_{T+1} = \frac{\bar{R}^{rf}_{T+1}}{1-\bar{a}^{rf}_T} \frac{\partial_{\sigma\sigma} a^{rf}_T - \partial_{\sigma\sigma} a^j_T}{2} - \mathbb{E}_T \partial_\sigma \mathrm{A}^k_{T+1} \partial_\sigma r^j_{T+1}.$$

(iii). Equations (11) and (12) hold.

Proof. (i). To the zeroth order, equation (11) is $\bar{x}_{T+1} = \frac{\beta \bar{M}_{T+1}}{\bar{M}_T} \frac{\bar{r}^j_{T+1}}{\bar{\xi}_{T+1}} = 0$. By lemma 7, $\frac{\beta \bar{M}_{T+1}}{\bar{M}_T} > 0$ and $\bar{\xi}_{T+1} < \infty$, therefore $\bar{r}^j_{T+1} = 0$. Since $\bar{r}^j_{T+1} = 0$, the first order expansion of (11) implies $\mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$. Apply these results to the zeroth and first order expansion of (9) to show that $\bar{a}^{rf}_T = \bar{a}^j_T, \partial_\sigma a^{rf}_T = \partial_\sigma a^j_T$. This shows part (i) of the lemma.

(ii). The second order expansion of (11) is

$$2\mathbb{E}_T \partial_\sigma \left(\frac{\beta M_{T+1}}{M_T}\right) \partial_\sigma r^j_{T+1} + 2\left(\overline{\frac{\beta M_{T+1}}{M_T}}\right) \frac{1}{\left(\overline{\xi^{-1}_{T+1}}\right)} \mathbb{E}_T \partial_\sigma \xi^{-1}_{T+1} \partial_\sigma r^j_{T+1} + \left(\overline{\frac{\beta M_{T+1}}{M_T}}\right) \mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1} = 0. \tag{32}$$

Applying these results to expansions of equation (9), we have

$$\partial_{\sigma\sigma} a^{rf}_T - \partial_{\sigma\sigma} a^j_T = 2\mathbb{E}_T \partial_\sigma \left(\frac{\beta M_{T+1}}{M_T}\right) \partial_\sigma r^j_{T+1} + \left(\overline{\frac{\beta M_{T+1}}{M_T}}\right) \mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1}.$$

Combine with (32) and observe that $\frac{\partial_\sigma \xi^{-1}_{T+1}}{\left(\overline{\xi^{-1}_{T+1}}\right)} = -\partial_\sigma \ln \xi_{T+1}$ and $\left(\overline{\frac{\beta M_{T+1}}{M_T}}\right) = \left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}}\right)$ to get

$$\partial_{\sigma\sigma} a^{rf}_T - \partial_{\sigma\sigma} a^j_T = 2\left(\overline{\frac{\beta M_{T+1}}{M_T}}\right) \mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r^j_{T+1} = 2\left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}}\right) \mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r^j_{T+1}. \tag{33}$$

This yields the first equation in part (ii).

When the government rolls over excess returns for additional k periods, the optimality condition reads

$$\mathbb{E}_T \frac{\beta M_{T+1}}{M_T} \frac{r^j_{T+1}}{\xi_{T+1+k}} \left[\left(\frac{\beta M_{T+2}}{M_{T+1}} R^{rf}_{T+2}\right) \times \ldots \left(\frac{\beta M_{T+1+k}}{M_{T+k}} R^{rf}_{T+1+k}\right)\right] = 0.$$

Expand the previous equation to get

$$\begin{aligned}
&\left(\overline{\frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}}}\right) \mathbb{E}_t \left[\left(\overline{\frac{\beta M_{t+1}}{M_t}}\right) \frac{1}{2} \partial_{\sigma\sigma} r^j_{t+1} + \partial_\sigma \left(\frac{\beta M_{t+1}}{M_t}\right) \partial_\sigma r^j_{t+1}\right. \\
&\left. + \left(\overline{\frac{\beta M_{t+1}}{M_t}}\right) \left(\overline{\frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}}}\right)^{-1} \partial_\sigma \left(\frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}}\right) \partial_\sigma r^j_{t+1}\right] \\
&= \left(\overline{\frac{\beta M_{t+1}}{M_t}}\right) \left(\overline{\frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}}}\right) \mathbb{E}_t \partial_\sigma \ln \xi_{t+1+k} \partial_\sigma r^j_{t+1}.
\end{aligned} \tag{34}$$

Define $\ln \tilde{A}^k_{t+1} = -\sum_{\tilde{t}=1}^{k} \ln\left(1 - a^{rf}_{t+1}\right)$. The optimality of the household from a rolling over a " synthetic" one period risk-free bond gives us

$$\begin{aligned} M_{t+1} &= \mathbb{E}_{t+1}\beta^k M_{t+1+k} R^{AAA,rf}_{t+2} \times ... \times R^{AAA,rf}_{t+1+k} \\ &= \mathbb{E}_{t+1}\beta^k M_{t+1+k} R^{rf}_{t+2} \times ... \times R^{rf}_{t+1+k} \times \frac{1}{1-a^{rf}_{t+1}} \times \frac{1}{1-a^{rf}_{t+2}} ... \frac{1}{1-a^{rf}_{t+k}}. \\ &= \mathbb{E}_{t+1}\beta^k M_{t+1+k} R^{rf}_{t+2} \times ... \times R^{rf}_{t+1+k} \times \tilde{A}^k_{t+1} \end{aligned}$$

Multiply by r^j_{t+1} and take expectations as of t :

$$\mathbb{E}_t r^j_{t+1} = \mathbb{E}_t r^j_{t+1} \frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}} \left[\tilde{A}^k_{t+1}\right].$$

Take second order expansions

$$\begin{aligned} \mathbb{E}_t \partial_{\sigma\sigma} r^j_{t+1} &= \left(\underbrace{\overline{\frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}} \left[\tilde{A}^k_{t+1}\right]}}_{=1} \right) \mathbb{E}_t \partial_{\sigma\sigma} r^j_{t+1} \\ &+ \left(\overline{\left[\tilde{A}^k_{t+1}\right]} \right) 2\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}} \\ &+ \left(\overline{\frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}}} \right) 2\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \left[\tilde{A}^k_{t+1}\right] \end{aligned}$$

Simplify to obtain

$$\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}} = -\left(\overline{\frac{\beta^k M_{t+1+k}}{M_{t+1} Q^{t+1+k}_{t+1}}} \right) \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \ln \tilde{A}^k_{t+1} \tag{35}$$

Subtitute (33) and (35) in (34) to get

$$\begin{aligned} &\left(\frac{1-\overline{a}^{rf}_t}{\overline{R}^{rf}_{t+1}} \right)^{-1} \mathbb{E}_t \left[\frac{\partial_{\sigma\sigma} a^{rf}_t - \partial_{\sigma\sigma} a^j_t}{2} \right] \\ = \;& \mathbb{E}_t \partial_\sigma \ln \xi_{t+1+k} \partial_\sigma r^j_{t+1} + \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \ln \tilde{A}^k_{t+1} \end{aligned}$$

Re-arrange to get the second equation in (ii).

(iii). Since

$$a^{rf}_t - a^j_t \simeq \frac{1}{2} \partial_{\sigma\sigma} \left(a^{rf}_t - a^j_t \right)$$

and

$$\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \ln \tilde{A}^k_{t+1} \simeq cov_T \left(\ln \mathrm{A}^k_{t+1}, r^j_{T+1} \right)$$

$$\frac{1-a^{rf}_T}{R^{rf}_{T+1}} cov_T \left(\ln \xi_{T+1}, r^j_{T+1} \right) \simeq \left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}} \right) \mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r^j_{T+1}$$

we obtain (11). Similar arguments yield (12). □

8.2 Proof of Theorem 1

We prove this theorem in several steps.

Lemma 9. *Equation (14) holds.*

Proof. Due to lemma 8(i), we have

$$\bar{\mathcal{Q}}_{T+1}^{T+t} = \frac{1}{\bar{R}_{T+2}^{rf}} \times ... \times \frac{1}{\bar{R}_{T+t}^{rf}} = \bar{Q}_{T+1}^{T+t}, \quad \mathbb{E}_T \partial_\sigma \mathcal{Q}_{T+1}^{T+t} = \mathbb{E}_T \partial_\sigma Q_{T+1}^{T+t} \text{ for } t > 1.$$

This implies that the first order terms in the expansion of (13) is (14). □

Lemma 10. *Equation (15) holds.*

Proof. The first order expansion of tax revenue elasticity is

$$\partial_\sigma \xi_t = -\bar{\gamma}_t \frac{1}{(1-\bar{\tau}_t)^2} \partial_\sigma \tau_t - \frac{\bar{\tau}_t}{1-\bar{\tau}_t} \partial_\sigma \gamma_t$$

and therefore

$$\partial_\sigma \ln \xi_t = -\frac{\bar{\gamma}_t}{\bar{\xi}_t (1-\bar{\tau}_t)^2} \left[\partial_\sigma \tau_t + \frac{\bar{\tau}_t (1-\bar{\tau}_t)}{\bar{\gamma}_t} \partial_\sigma \gamma_t \right].$$

Let $\mathcal{R}(\tau_t, \gamma_t, \theta_t) \equiv \tau_t Y_t$ be tax revenues, where $Y_t = (1-\tau_t)^{\gamma_t} \theta_t^{1+\gamma_t}$. Let $\bar{\mathcal{R}}_{\tau,t}, \bar{\mathcal{R}}_{\gamma,t}, \bar{\mathcal{R}}_{\theta,t}$ be the derivatives of this function evaluated at $\left(\bar{\tau}_t, \bar{\gamma}_t, \bar{\theta}_t\right)$. We have $\frac{\partial \mathcal{R}}{\partial \tau} = \frac{\partial \ln \mathcal{R}}{\partial \ln \tau} \frac{\tau Y}{\tau} = \xi Y$. Therefore we can write

$$\begin{aligned} \partial_\sigma \mathcal{R}(\tau_t, \gamma_t, \theta_t) &= \bar{\xi}_t \bar{Y}_t \partial_\sigma \tau_t + \bar{\mathcal{R}}_{\gamma,t} \partial_\sigma \gamma_t + \bar{\mathcal{R}}_{\theta,t} \partial_\sigma \theta_t \\ &= \bar{\xi}_t \bar{Y}_t \left[\partial_\sigma \tau_t + \frac{\bar{\tau}_t (1-\bar{\tau}_t)}{\bar{\gamma}_t} \partial_\sigma \gamma_t \right] + \left[\bar{\mathcal{R}}_{\gamma,t} - \bar{\xi}_t \bar{Y}_t \frac{\bar{\tau}_t (1-\bar{\tau}_t)}{\bar{\gamma}_t} \right] \partial_\sigma \gamma_t + \bar{\mathcal{R}}_{\theta,t} \partial_\sigma \theta_t \\ &= -\frac{\bar{\xi}_t^2 (1-\bar{\tau}_t)^2 \bar{Y}_t}{\bar{\gamma}_t} \partial_\sigma \ln \xi_t + \left\{ \left[\bar{\mathcal{R}}_{\gamma,t} - \bar{\xi}_t \bar{Y}_t \frac{\bar{\tau}_t (1-\bar{\tau}_t)}{\bar{\gamma}_t} \right] \partial_\sigma \gamma_t + \bar{\mathcal{R}}_{\theta,t} \partial_\sigma \theta_t \right\}. \end{aligned}$$

Primary deficit is $X_t = G_t - \mathcal{R}(\tau_t, \gamma_t, \theta_t)$ and, therefore,

$$\partial X_t = \frac{\bar{\xi}_t^2 (1-\bar{\tau}_t)^2 \bar{Y}_t}{\bar{\gamma}_t} \partial_\sigma \ln \xi_t + \underbrace{\left\{ \partial_\sigma G_t - \left[\bar{\mathcal{R}}_{\gamma,t} - \bar{\xi}_t \bar{Y}_t \frac{\bar{\tau}_t (1-\bar{\tau}_t)}{\bar{\gamma}_t} \right] \partial_\sigma \gamma_t - \bar{\mathcal{R}}_{\theta,t} \partial_\sigma \theta_t \right\}}_{\equiv \partial_\sigma X_t^\perp}.$$

Routine algebra shows that

$$\zeta_t \equiv \mathbb{E}_T \xi_t^2 \frac{(1-\tau_t)^2}{\gamma_t} = \mathbb{E}_T \frac{(1-\tau_t - \tau_t \gamma_t)^2}{\gamma_t},$$

which verifies equation (15). □

Lemma 11. *(i). Let $w_{T+t} \equiv q_T^{rf} Q_{T+1}^{T+t} \frac{Y_{T+t}}{Y_T}$ for $t \geq 1$. The optimal portfolio satisfies*

$$\sum_{t=1}^{\infty} \bar{w}_{T+t} \left[\frac{\bar{q}_{T+t}^{rf} \overline{X}_{T+1+t}}{\bar{Y}_{T+t}} \right] \mathbb{E}_T \partial_\sigma r_{T+1}^j \partial_\sigma \ln Q_{T+1}^{T+1+t} \tag{36}$$

$$+ \sum_{t=1}^{\infty} \bar{w}_{T+t} \mathbb{E}_T \partial_\sigma r_{T+1}^j \frac{\partial_\sigma X_{T+t}^{\perp}}{\bar{Y}_{T+t}}$$

$$+ \frac{1}{1 - \bar{a}_T^{rf}} \left(\frac{1}{\bar{q}_T^{rf}} \sum_{t=1}^{\infty} \bar{w}_{T+t} \bar{\zeta}_{T+t} \right) \left(\frac{\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j}{2} \right) \tag{37}$$

$$- \bar{q}_T^{rf} \sum_{t=1}^{\infty} \bar{w}_{T+t} \left[\frac{\bar{q}_{T+t}^{rf}}{\bar{q}_T^{rf}} \frac{\bar{Y}_{T+1+t}}{\bar{Y}_{T+t}} \bar{\zeta}_{T+1+t} \right] \mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^t \partial_\sigma r_{T+1}^j$$

$$= \left(\sum_{i \geq 1} \mathbb{E}_T \partial_\sigma r_{T+1}^i \partial_\sigma r_{T+1}^j \overline{\omega}_T^i \right) \bar{q}_T^{rf} \frac{\overline{B}_T}{\bar{Y}_T}.$$

(ii). If stationarity conditions (17) hold then $\bar{w}_{T+t} = (q\Gamma)^t$ and equation (36) becomes

$$(1 - q\Gamma) \frac{\bar{B}_T}{\bar{Y}_T} \sum_{t=1}^{\infty} (q\Gamma)^t \mathbb{E}_T \partial_\sigma r_{T+1}^j \partial_\sigma \ln Q_{T+1}^{T+1+t} + \sum_{t=1}^{\infty} (q\Gamma)^t \mathbb{E}_T \partial_\sigma r_{T+1}^j \frac{\partial_\sigma X_{T+t}^{\perp}}{\bar{Y}_{T+t}}$$

$$- \bar{\zeta}_T \left[(q\Gamma) \sum_{t=1}^{\infty} (q\Gamma)^t \mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^t \partial_\sigma r_{T+1}^j \right]$$

$$= \left(\sum_{i \geq 1} \mathbb{E}_T \partial_\sigma r_{T+1}^i \partial_\sigma r_{T+1}^j \overline{\omega}_T^i \right) q \frac{\overline{B}_T}{\bar{Y}_T}.$$

Proof. (i). Multiply (14) by $\frac{\partial_\sigma r_{T+1}^j}{\bar{Y}_T}$, take expectation at time T, use the fact that $R_{T+1}^{rf} B_T$ and Q_{T+1}^{T+1} are time-T measurable, that $\mathbb{E}_T \partial_\sigma r_{T+1}^j = 0$ by lemma 8(i), and that $\mathbb{E}_T \partial_\sigma \ln x_{T+t} = \mathbb{E}_T \frac{\partial_\sigma x_{T+t}}{\bar{x}_{T+t}}$ for any variable x_{T+t} to obtain

$$\sum_{t=2}^{\infty} \overline{Q}_{T+1}^{T+t} \overline{X}_{T+t} \mathbb{E}_T \partial_\sigma \ln Q_{T+1}^{T+t} \partial_\sigma r_{T+1}^j + \sum_{t=1}^{\infty} \overline{Q}_{T+1}^{T+t} \mathbb{E}_{T+1} \partial_\sigma X_{T+t} \partial_\sigma r_{T+1}^j = \left(\sum_{i \geq 1} \partial_\sigma r_{T+1}^i \partial_\sigma r_{T+1}^j \overline{\omega}_T^i \right) \overline{B}_T.$$

Substitute (15):

$$\sum_{t=2}^{\infty} \overline{Q}_{T+1}^{T+t} \frac{\bar{Y}_{T+t}}{\bar{Y}_T} \frac{\overline{X}_{T+t}}{\bar{Y}_{T+t}} \mathbb{E}_T \partial_\sigma \ln Q_{T+1}^{T+t} \partial_\sigma r_{T+1}^j + \sum_{t=1}^{\infty} \overline{Q}_{T+1}^{T+t} \frac{\bar{Y}_{T+t}}{\bar{Y}_T} \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^{\perp}}{\bar{Y}_{T+t}} \partial_\sigma r_{T+1}^j$$

$$+ \left(\frac{\bar{R}_{T+1}^{rf}}{1 - \bar{a}_T^{rf}} \right) \left(\sum_{t=1}^{\infty} \overline{Q}_{T+1}^{T+t} \frac{\bar{Y}_{T+t}}{\bar{Y}_T} \bar{\zeta}_{T+t} \right) \frac{\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j}{2}$$

$$- \sum_{t=2}^{\infty} \overline{Q}_{T+1}^{T+t} \frac{\bar{Y}_{T+t}}{\bar{Y}_T} \bar{\zeta}_{T+t} \mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^{t-1} \partial_\sigma r_{T+1}^j$$

$$= \left(\sum_{i \geq 1} \mathbb{E}_T \partial_\sigma r_{T+1}^i \partial_\sigma r_{T+1}^j \overline{\omega}_T^i \right) \frac{\overline{B}_T}{\bar{Y}_T}.$$

Multiply both sides by $\bar{q}_T^{rf}$, re-arrange indices so that all summations start with $t = 1$ and use definition of w_{T+t} to obtain equation (36).

(ii). Stationarity conditions (17) imply that

$$\bar{\xi}_{T+t} = \bar{\xi}_T, \quad \bar{\zeta}_{T+t} = \bar{\zeta}_T, \quad \frac{\bar{Y}_{T+t}}{\bar{Y}_T} = \Gamma^t, \quad \bar{q}_T^{rf} = q, \quad \overline{Q}_{T+1}^{T+t} = q^{t-1}, \quad \bar{w}_t = (q\Gamma)^t, \quad \frac{\bar{X}_{T+t}}{\bar{Y}_{T+t}} = \frac{\bar{X}_T}{\bar{Y}_T}.$$

Therefore, equation (36) becomes

$$\begin{aligned} & q\Gamma \frac{\bar{X}_T}{\bar{Y}_T} \sum_{t=1}^{\infty} (q\Gamma)^t \, \mathbb{E}_T \partial_\sigma \ln Q_{T+1}^{T+1+t} \partial_\sigma r_{T+1}^j + \sum_{t=1}^{\infty} (q\Gamma)^t \, \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^{\perp}}{\bar{Y}_{T+t}} \partial_\sigma r_{T+1}^j \\ & + \frac{\bar{\zeta}_T}{1-\bar{a}_T^{rf}} \left(\frac{1}{q} \sum_{t=1}^{\infty} (q\Gamma)^t \right) \frac{\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j}{2} - (q\Gamma) \sum_{t=1}^{\infty} (q\Gamma)^t \, \mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^t \partial_\sigma r_{T+1}^j \\ = \; & \left(\sum_{i \geq 1} \mathbb{E}_T \partial_\sigma r_{T+1}^i \partial_\sigma r_{T+1}^j \overline{\omega}_T^i \right) q \frac{\overline{B}_T}{\bar{Y}_T}. \end{aligned}$$

The zeroth order government budget constraint can be written as $\sum_{t=1}^{\infty} q^t \frac{\bar{Y}_{T+t}}{\bar{Y}_T} \frac{\bar{X}_{T+t}}{\bar{Y}_{T+t}} = \frac{\bar{B}_T}{\bar{Y}_T}$. Applying stationarity conditions, we obtain $q\Gamma \frac{\bar{X}_T}{\bar{Y}_T} = (1 - q\Gamma) \frac{\bar{B}_T}{\bar{Y}_T}$. Substitute this into the previous equation to show part (ii) of this lemma. □

Lemma 12. *Theorem 1 holds.*

Proof. Consider any three equilibrium variables $x_{T+1}, z_{T+1}, \varrho_{T+1}$ where ϱ_{T+1} satisfies

$$\bar{\varrho}_{T+1} = \mathbb{E}_T \partial_\sigma \bar{\varrho}_{T+1} = 0. \tag{38}$$

We have

$$\mathbb{E}_T x_{T+1} \varrho_{T+1} \simeq \mathbb{E}_T \left[\partial_\sigma x_{T+1} \partial_\sigma \varrho_{T+1} + \frac{1}{2} \bar{x}_{T+1} \partial_{\sigma\sigma} \varrho_{T+1} \right], \; \mathbb{E}_T x_{T+1} \mathbb{E}_T \varrho_{T+1} \simeq \mathbb{E}_T \left[\frac{1}{2} \bar{x}_{T+1} \partial_{\sigma\sigma} \varrho_{T+1} \right],$$

and, therefore,

$$\mathbb{E}_T z_{T+1} cov_T \left(x_{T+1}, \varrho_{T+1} \right) = \mathbb{E}_T z_{T+1} \times \left[\mathbb{E}_T x_{T+1} \varrho_{T+1} - \mathbb{E}_T x_{T+1} \mathbb{E}_T \varrho_{T+1} \right] \simeq \bar{z}_{T+1} \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma \varrho_{T+1}.$$

Since both $\bar{r}_{T+1}^j$ and $a_T^{rf} - a_T^j$ satisfy property (38) due to lemma 8(i), equation (36) can be re-written as

$$\sum_{t=1}^{\infty}\mathbb{E}_T w_{T+t}\times\mathbb{E}_T\frac{q^{rf}_{T+t}X_{T+1+t}}{Y_{T+t+1}}\times cov_T\left(\ln Q^{T+1+t}_{T+1},r^j_{T+1}\right)+\sum_{t=1}^{\infty}\mathbb{E}_T w_{T+t}\times cov_T\left(\frac{X^{\perp}_{T+t}}{\mathbb{E}_T Y_{T+t}},r^j_{T+1}\right)$$
$$+\left(\sum_{t=1}^{\infty}\mathbb{E}_T w_{T+t}\times\frac{\mathbb{E}_T\zeta_{T+t}}{\left(1-a^{rf}_T\right)q^{rf}_T}\right)\left(a^{rf}_T-a^j_T\right)$$
$$-\sum_{t=1}^{\infty}\mathbb{E}_T w_{T+t}\times\mathbb{E}_T q^{rf}_{T+t}\frac{Y_{T+1+t}}{Y_{T+t}}\zeta_{T+1+t}\times cov_T\left(\mathrm{A}^t_{T+1},r^j_{T+1}\right)$$
$$=\left(\sum_{i\geq 1}cov_T\left(r^i_{T+1},r^j_{T+1}\right)\omega^i_T\right)q^{rf}_T\frac{B_T}{Y_T}.$$

Written in the matrix form, this is equation (16). Same arguments applied to the equation shown in lemma 11(ii) establishes (18). □

Corollary 2 follows from the following lemma.

Lemma 13. *Let $q^{(t)}_T, r^{(t)}_T$ be the period-T price and excess return of a pure discount bond that expires in period t. Then $q^{(T+1)}_T cov_T\left(r^{(T+1+t)}_{T+1},r^j_{T+1}\right)\simeq cov_T\left(Q^{T+1+t}_{T+1},r^j_{T+1}\right)$ for any security j that the government can trade. In particular, if the government can only trade pure discount bonds of all maturities and matrix Σ_T is arranged so that its i^{th} column corresponds to bonds expiring in period $T+i$, then $q^{rf}_T\Sigma_T\simeq\Sigma^Q_T$.*

Proof. We show that

$$\bar{q}^{(T+1)}_T\mathbb{E}_T\partial_\sigma r^{(T+1+t)}_{T+1}\partial_\sigma r^j_{T+1}=\mathbb{E}_T\partial_\sigma\ln Q^{T+1+t}_{T+1}\partial_\sigma r^j_{T+1},\tag{39}$$

which is equivalent to $\bar{q}^{rf}_T\mathbb{E}_T\partial_\sigma r^i_{T+1}\partial_\sigma r^j_{T+1}=\mathbb{E}_T\partial_\sigma\ln Q^{T+1+t}_{T+1}\partial_\sigma r^j_{T+1}$ in the notation used in body of the paper. The latter equation implies that $q^{rf}_T\Sigma_T\simeq\Sigma^Q_T$ under the conditions stated in this lemma due to the same arguments that were used in the proof of lemma 12.

Step 1. $\bar{q}^{(T+1)}_T\mathbb{E}_T\partial_\sigma r^{(T+1+t)}_{T+1}\partial_\sigma r^j_{T+1}=\mathbb{E}_T\left[\partial_\sigma\ln\frac{\beta^t M_{T+1+t}}{M_{T+1}}-\sum_{k=1}^t\partial_\sigma\ln\left(1-a^{(T+1+t)}_{T+k}\right)\right]\partial_\sigma r^j_{T+1}$.

The definition of returns and liquidity premium imply that

$$\begin{aligned}q^{(T+1+t)}_T &= \mathbb{E}_T\frac{\beta M_{T+1}}{M_T}q^{(T+1+t)}_{T+1}\frac{1}{1-a^{(T+1+t)}_T}\\ &= \mathbb{E}_T\left[\frac{\beta^{1+t}M_{T+1+t}}{M_T}\frac{1}{1-a^{(T+1+t)}_T}\times\ldots\times\frac{1}{1-a^{(T+1+t)}_{T+t}}\right].\end{aligned}$$

Therefore, the excess return is

$$
\begin{aligned}
r_{T+1}^{(T+1+t)} &= \frac{q_{T+1}^{(T+1+t)}}{q_T^{(T+1+t)}} - \frac{1}{q_T^{(T+1)}} \\
&= \frac{1}{\frac{1}{1-a_T^{(T+1+t)}}} \frac{\mathbb{E}_{T+1}\left[\frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+1+t)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}}\right]}{\mathbb{E}_T\left[\frac{\beta M_{T+1}}{M_T} \frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+1+t)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}}\right]} - \frac{1}{\mathbb{E}_T\left[\frac{\beta M_{T+1}}{M_T} \frac{1}{1-a_T^{(T+1)}}\right]}.
\end{aligned}
$$

Its first order approximation terms can be written as

$$
\begin{aligned}
\partial_\sigma r_{T+1}^{(T+1+t)} &= \left(1-\bar{a}_T^{(T+1+t)}\right) \frac{\bar{M}_T}{\beta \bar{M}_{T+1}} \frac{\mathbb{E}_{T+1}\partial_\sigma\left[\frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+1+t)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}}\right]}{\overline{\left[\frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+1+t)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}}\right]}} + \text{t.m.}T \\
&= \frac{\bar{M}_T}{\beta \bar{M}_{T+1}} \frac{1}{1-\bar{a}_T^{(T+1+t)}} \mathbb{E}_{T+1}\left[\partial_\sigma \ln \frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^{t} \partial_\sigma \ln\left(1-a_{T+k}^{(T+1+t)}\right)\right] + \text{t.m.}T,
\end{aligned}
$$

where "t.m.T" denotes "terms measurable with respect to time T". Since $\bar{a}_T^{(T+1+t)} = \bar{a}_T^{(T+1)}$ and $\mathbb{E}_T \partial_\sigma r_{T+1}^j = 0$ for any j by lemma 8(i), this equation imply that

$$
\mathbb{E}_T \partial_\sigma r_{T+1}^{(T+1+t)} \partial_\sigma r_{T+1}^j = \frac{1}{\bar{q}_T^{(T+1)}} \mathbb{E}_{T+1}\left[\partial_\sigma \ln \frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^{t} \partial_\sigma \ln\left(1-a_{T+k}^{(T+1+t)}\right)\right] \partial_\sigma r_{T+1}^j.
$$

This proves Step 1.

Step 2. $\mathbb{E}_T \partial_\sigma \ln Q_{T+1}^{T+1+t} \partial_\sigma r_{T+1}^j = \mathbb{E}_T\left[\partial_\sigma \ln \frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^{t} \partial_\sigma \ln\left(1-a_{T+k}^{(T+1+k)}\right)\right] \partial_\sigma r_{T+1}^j$.

By definition of Q_{T+1}^{T+1+t} we have

$$
\begin{aligned}
Q_{T+1}^{T+1+t} &= \mathbb{E}_{T+1} \frac{\beta M_{T+2}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+2)}} \times \mathbb{E}_{T+2} \frac{\beta M_{T+3}}{M_{T+2}} \frac{1}{1-a_{T+2}^{(T+3)}} \times ... \times \mathbb{E}_{T+t} \frac{\beta M_{T+1+t}}{M_{T+t}} \frac{1}{1-a_{T+t}^{(T+1+t)}} \\
&= \mathbb{E}_{T+1} \frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+2)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}_T \partial_\sigma \ln Q_{T+1}^{T+1+t} \partial_\sigma r_{T+1}^j &= \frac{\mathbb{E}_T \partial_\sigma \left[\frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+2)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}}\right] \partial r_{T+1}^j}{\overline{\left[\frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+2)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}}\right]}} \\
&= \mathbb{E}_T\left[\partial_\sigma \ln \frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^{t} \partial_\sigma \ln\left(1-a_{T+k}^{(T+1+k)}\right)\right] \partial_\sigma r_{T+1}^j.
\end{aligned}
$$

Step 3. Equation (39) holds.

Note that

$$\mathbb{E}_T\partial_\sigma \ln\left(1-a_{T+k}^{(T+1+t)}\right)\partial_\sigma r_{T+1}^j = \mathbb{E}_T\left\{\mathbb{E}_{T+k}\partial_\sigma \ln\left(1-a_{T+k}^{(T+1+t)}\right)\right\}\partial_\sigma r_{T+1}^j$$
$$= \mathbb{E}_T\left\{\mathbb{E}_{T+k}\partial_\sigma \ln\left(1-a_{T+k}^{(T+1+k)}\right)\right\}\partial_\sigma r_{T+1}^j = \mathbb{E}_T\partial_\sigma \ln\left(1-a_{T+k}^{(T+1+k)}\right)\partial_\sigma r_{T+1}^j,$$

where we applied the law of iterated expectations in the first and third equations, and lemma 8(i) in the second equations. This implies that the right hand sides of equations obtained in Step 1 and 2 are the same, proving (39). □

9 Proofs for section 6

Lemma 14. *Equation (27) holds*

Proof. Direct computations of equation (6) and (7) for the one-time perturbation yields

$$\begin{aligned}
&-\frac{\beta^T M_T\left(s^T\right)}{\xi_T\left(s^T\right)}\left[\partial_\epsilon q_T^j\left(s^T\right)\left(B_T^j\left(s^T\right)-B_{T-1}^j\left(s^{T-1}\right)\right)-\partial_\epsilon q_T^{rf}\left(s^T\right)\left(B_T^{rf}\left(s^T\right)-B_{T-1}^{rf}\left(s^T\right)\right)\right]\\
&-\beta^T M_T\left(s^T\right)\left[\partial_\epsilon q_T^j\left(s^T\right)\left(b_T^j\left(s^T\right)-b_{T-1}^j\left(s^{T-1}\right)\right)-\partial_\epsilon q_T^{rf}\left(s^T\right)\left(b_T^{rf}\left(s^T\right)-b_{T-1}^{rf}\left(s^T\right)\right)\right]\\
&+\beta^T\sum_{i=\{j,rf\}}\frac{\partial V_0}{\partial V_1\left(s^1\right)}\times\ldots\times\frac{\partial V_{T-1}\left(s^{T-1}\right)}{\partial V_T\left(s^T\right)}\frac{\partial U_T\left(s^T\right)}{\partial\left(q_T^i b_T^i\right)}b_T^i\left(s^T\right)\partial_\epsilon q_T^i\left(s^T\right)\\
&+\beta^{T+1}\sum_{s^{T+1}}M_{T+1}\left(s^{T+1}\right)\frac{r_{T+1}^j\left(s^{T+1}\right)}{\xi_{T+1}\left(s^{T+1}\right)}\\
=\;& 0.
\end{aligned}\tag{40}$$

To simplify this equation, observe that households optimality condition for asset i is

$$\begin{aligned}
&-\frac{\partial V_0}{\partial V_1\left(s^1\right)}\times\ldots\times\frac{\partial V_{T-1}\left(s^{T-1}\right)}{\partial V_T\left(s^T\right)}\frac{\partial U_T\left(s^T\right)}{\partial\left(q_T^i b_T^i\right)}q_T^i\left(s^T\right)+\Pr\left(s^T\right)M_T\left(s^T\right)q_T^i\left(s^T\right)\\
=\;&\sum_{s^{T+1}}\Pr\left(s^{T+1}\right)\beta M\left(s^{T+1}\right)\left[q_{T+1}^i\left(s^{T+1}\right)+d_{T+1}^i\left(s^{T+1}\right)\right],
\end{aligned}$$

which can be written as

$$1-\underbrace{\frac{\partial V_0}{\partial V_1\left(s^1\right)}\times\ldots\times\frac{\partial V_{T-1}\left(s^{T-1}\right)}{\partial V_T\left(s^T\right)}\frac{\partial U_T\left(s^T\right)/\partial\left(q_T^i b_T^i\right)}{\Pr\left(s^T\right)M\left(s^T\right)}}_{=a_T^i\left(s^T\right).}=\sum_{s^{T+1}}\Pr\left(s^{T+1}|s^T\right)\frac{\beta M_{T+1}\left(s^{T+1}\right)}{M_T\left(s^T\right)}R_{T+1}^i\left(s^{T+1}\right).$$

Our perturbation has $\partial_\epsilon B_T^j=\frac{1}{q_T^j}$ and $\partial_\epsilon B_T^{rf}=-\frac{1}{q_T^{rf}}$. From (25), we get that $\partial_\epsilon q_T^j=\lambda_T^j$ and

$\partial_\epsilon q^{rf}_T = -\lambda^{rf}_T$. Therefore we can re-write (40) as

$$
\begin{aligned}
&\frac{1}{\xi_T\left(s^T\right)}\left[\lambda^j_T\left(s^T\right)\left(B^j_T\left(s^T\right)-B^j_{T-1}\left(s^{T-1}\right)\right)-\lambda^{rf}_T\left(s^T\right)\left(B^{rf}_T\left(s^T\right)-B^{rf}_{T-1}\left(s^T\right)\right)\right]\\
&+\left[\lambda^j_T\left(s^T\right)\left(b^j_T\left(s^T\right)-b^j_{T-1}\left(s^{T-1}\right)\right)-\lambda^{rf}_T\left(s^T\right)\left(b^{rf}_T\left(s^T\right)-b^{rf}_{T-1}\left(s^T\right)\right)\right]\\
&-a^j_T b^j_T\left(s^T\right)\lambda^j_T\left(s^T\right)+a^{rf}_T b^{rf}_T\left(s^T\right)\lambda^{rf}_T\left(s^T\right)\\
&-\sum_{s^{T+1}}\frac{\beta M_{T+1}\left(s^{T+1}\right)}{M_T\left(s^T\right)}\frac{r^j_{T+1}\left(s^{T+1}\right)}{\xi_{T+1}\left(s^{T+1}\right)}\\
&=0.
\end{aligned}
$$

Given then definition of ϕ^i_t, this becomes

$$
\begin{aligned}
\mathbb{E}_T\frac{\beta M_{T+1}}{M_T}\frac{r^j_{T+1}}{\xi_{T+1}} \quad = \quad & \left\{\lambda^j_T\left(\frac{1}{\xi_T}-\phi^j_T+a^j_T\phi^j_T\right)B^j_T-\lambda^j_T\left(\frac{1}{\xi_T}-\phi^j_{T-1}\right)B^j_{T-1}\right\}\\
&-\left\{\lambda^{rf}_T\left(\frac{1}{\xi_T}-\phi^{rf}_T+a^{rf}_T\phi^{rf}_T\right)B^{rf}_T-\lambda^{rf}_T\left(\frac{1}{\xi_T}-\phi^{rf}_{T-1}\right)B^{rf}_{T-1}\right\}.
\end{aligned}
$$

□

Lemma 15. *Equation (27) holds*

Proof. The generalized optimality condition (28) for k period rollover is

$$
\begin{aligned}
&\left[cov_T\left(\ln\xi_{T+1+k}, r^j_{T+1}\right)+cov_T\left(\mathrm{A}^k_{T+1}, r^j_{T+1}\right)\right]\\
\simeq\ &\left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}}\right)^{-1}\left(a^{rf}_T-a^j_T\right)\\
&-\frac{\overline{\xi_{T+1+k}}}{2}\left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}}\right)^{-1}\left\{\lambda^j_T\left(\frac{1}{\bar{\xi}_T}-\overline{\phi}^j_T+\overline{a}^j_T\overline{\phi}^j_T\right)\overline{B}^j_T-\lambda^j_T\left(\frac{1}{\bar{\xi}_T}-\overline{\phi}^j_{T-1}\right)\overline{B}^j_{T-1}\right\}\\
&+\frac{\overline{\xi_{T+1+k}}}{2}\left(\overline{\frac{1-a^{rf}_T}{R^{rf}_{T+1}}}\right)^{-1}\left\{\lambda^{rf}_T\left(\frac{1}{\bar{\xi}_T}-\overline{\phi}^{rf}_T+\overline{a}^{rf}_T\overline{\phi}^{rf}_T\right)\overline{B}^{rf}_T-\lambda^{rf}_T\left(\frac{1}{\bar{\xi}_T}-\overline{\phi}^{rf}_{T-1}\right)\overline{B}^{rf}_{T-1}\right\}.
\end{aligned}
$$

Under the assumptions $\overline{\phi}^j_T = \phi; \overline{a}^j_T = \overline{a}^{rf}_T$ and $\lambda^{rf}_T = 0$ we get that

$$\begin{aligned}
&\left\{\lambda^j_T\left(\frac{1}{\overline{\xi}_T} - \phi + \overline{a}_T\phi\right)\overline{B}^j_T - \lambda^j_T\left(\frac{1}{\overline{\xi}_T} - \phi\right)\overline{B}^j_{T-1}\right\}\\
&= \frac{\overline{B}_T}{\overline{Y}_T}\left\{\frac{\overline{Y}_T\lambda^j_T}{\overline{q}^j_T}\left(\frac{1}{\overline{\xi}_T} - \phi + \overline{a}_T\phi\right)\frac{\overline{q}^j_T\overline{B}^j_T}{\overline{B}_T} - \frac{\overline{Y}_T\lambda^j_T}{\overline{q}^j_T}\left(\frac{1}{\overline{\xi}_T} - \phi\right)\underbrace{\frac{\overline{q}^j_{T-1}\overline{B}^j_{T-1}}{\overline{B}_{T-1}} \times \frac{\overline{q}^j_T}{\overline{q}^j_{T-1}}\frac{\overline{B}_{T-1}}{\overline{B}_T}}_{\overline{\hat{\omega}}^j_{T-1}}\right\}\\
&= \frac{\overline{B}_T}{\overline{Y}_T}\left\{\frac{\overline{Y}_T\lambda^j_T}{\overline{q}^j_T}\left(\frac{1}{\overline{\xi}_T} - \phi + \overline{a}_T\phi\right)\overline{\omega}^j_T - \frac{\overline{Y}_T\lambda^j_T}{\overline{q}^j_T}\left(\frac{1}{\overline{\xi}_T} - \phi\right)\overline{\hat{\omega}}^j_{T-1}\right\}\\
&= \frac{\overline{B}_T}{\overline{Y}_T}\left\{\frac{\overline{Y}_T\lambda^j_T}{\overline{q}^j_T}\left(\frac{1}{\overline{\xi}_T} - \phi\right)\left(\overline{\omega}^j_T - \overline{\hat{\omega}}^j_{T-1}\right) + \frac{\overline{Y}_T\lambda^j_T}{\overline{q}^j_T}\left(\overline{a}_T\phi\right)\overline{\omega}^j_T\right\}
\end{aligned}$$

Define matrix Λ_T

$$\Lambda_T = \begin{bmatrix} \frac{\overline{Y}_T\lambda^1_T}{\overline{q}^1_T} & & 0 & ... \\ & & \frac{\overline{Y}_T\lambda^2_T}{\overline{q}^2_T} & ... \\ ... & ... & ... & ... \end{bmatrix}$$

Stack up for all j to get

$$\begin{aligned}
cov_T\left(\ln\xi_{T+1+k}, \boldsymbol{r}_{T+1}\right) + cov_T\left(\mathrm{A}^k_{T+1}, \boldsymbol{r}_{T+1}\right) \quad \simeq \quad & -\left(\frac{\overline{1 - a^{rf}_T}}{R^{rf}_{T+1}}\right)^{-1}\left[\boldsymbol{a}_T - \mathbf{1}\cdot a^{rf}_T\right]\\
& -\left(\frac{\overline{\xi_{T+1+k}}}{2}\right)\left(\frac{\overline{1 - a^{rf}_T}}{R^{rf}_{T+1}}\right)^{-1}\left(\frac{\overline{B}_T}{\overline{Y}_T}\right)\left(\frac{1}{\overline{\xi}_T} - \phi\right)\Lambda_T\left(\boldsymbol{\omega}_T - \hat{\boldsymbol{\omega}}_{T-}\right.\\
& -\left(\frac{\overline{\xi_{T+1+k}}}{2}\right)\left(\frac{\overline{1 - a^{rf}_T}}{R^{rf}_{T+1}}\right)^{-1}\left(\frac{\overline{B}_T}{\overline{Y}_T}\right)\left(\overline{a}_T\phi\right)\Lambda_T\boldsymbol{\omega}_T
\end{aligned}$$

Substitute in the budget constraint to get

$$\sum_{t=1}^{\infty} \mathbb{E}_T w_{T+t} \times \mathbb{E}_T \frac{q^{rf}_{T+t} X_{T+1+t}}{Y_{T+t}} \times cov_T \left(\ln Q^{T+1+t}_{T+1}, r^j_{T+1} \right) + \sum_{t=1}^{\infty} \mathbb{E}_T w_{T+t} \times cov_T \left(\frac{X^{\perp}_{T+t}}{\mathbb{E}_T Y_{T+t}}, r^j_{T+1} \right)$$
$$- \left(\sum_{t=1}^{\infty} \mathbb{E}_T w_{T+t} \times \frac{\mathbb{E}_T \zeta_{T+t}}{\left(1 - a^{rf}_T\right) q^{rf}_T} \right) \left(a^j_T - a^{rf}_T \right)$$
$$- \sum_{t=1}^{\infty} \mathbb{E}_T w_{T+t} \times \mathbb{E}_T q^{rf}_{T+t} \frac{Y_{T+1+t}}{Y_{T+t}} \zeta_{T+1+t} \times cov_T \left(\mathrm{A}^t_{T+1}, r^j_{T+1} \right)$$
$$- \left(\frac{q^{rf}_T B_T}{\bar{Y}_T} \right) \underbrace{\left(\frac{\overline{1 - a^{rf}_T}}{R^{rf}_{T+1}} \right)^{-1} \frac{1}{q^{rf}_T} \left(\sum_{t=1}^{\infty} \mathbb{E}_T w_{T+t} \times \mathbb{E}_T \zeta_{T+t} \xi_{T+t} \right)}_{\chi_T > 0} \frac{\overline{Y}_T \lambda^j_T}{\overline{q}^j_T} \left[\left(\frac{1}{\bar{\xi}_T} - \phi \right) \left(\omega^j_T - \hat{\boldsymbol{\omega}}^j_{T-1} \right) + \left(\overline{a}_T \phi \right) \boldsymbol{\omega}^j_T \right]$$
$$\simeq \quad \left(\sum_{i \geq 1} cov_T \left(r^i_{T+1}, r^j_{T+1} \right) \omega^i_T \right) q^{rf}_T \frac{B_T}{Y_T}.$$

In matrix form,

$$q^{rf}_T \frac{B_T}{Y_T} \Sigma_T \boldsymbol{\omega}_T \simeq \left[\Sigma^Q_T \Pi^Q_T + \Sigma^X_T + \left(\Sigma^a_T \Pi^a_T - \Sigma^A_T \Pi^A_T \right) \right] \mathbf{w}_T.$$
$$- \left(\frac{q^{rf}_T B_T}{\bar{Y}_T} \right) \chi_T \Lambda_T \left[\left(\frac{1}{\bar{\xi}_T} - \phi \right) \left(\omega^j_T - \hat{\boldsymbol{\omega}}^j_{T-1} \right) + \left(\overline{a}_T \phi \right) \boldsymbol{\omega}^j_T \right]$$

When the market structure is zero coupon bonds of all maturities, we have shown that $q^{rf}_T \Sigma_T \simeq \Sigma^Q_T$. The zeroth order version of (26) means that

$$0 \quad = \quad \left\{ \left(\frac{1}{\bar{\xi}_T} - 1 \right) - \frac{\beta \overline{M}_{T+1}}{\overline{M}_T} \overline{R}^j_{T+1} \left(\frac{1}{\bar{\xi}_{T+1}} - 1 \right) \right\} + \overline{a}^j_T$$

Stationary taxes means that $\bar{\xi}_T = \bar{\xi}_{T+1}$ and the previous equation is

$$0 \quad = \left(\tfrac{1}{\bar{\xi}_T} - 1 \right) \quad \left\{ 1 - \frac{\beta \overline{M}_{T+1}}{\overline{M}_T} \overline{R}^j_{T+1} \right\} + \overline{a}^j_T.$$

The zeroth order version of (9) says that $1 - \frac{\beta \overline{M}_{T+1}}{\overline{M}_T} \overline{R}^j_{T+1} = \overline{a}^j_T$ and therefore we get that

$$0 = \frac{\overline{a}^j_T}{\bar{\xi}_T} \implies \overline{a}_T = 0.$$

Following the same steps as in part (ii) of the proof of Lemma (11), we get

$$\boldsymbol{\omega}_T \simeq (1 - q\Gamma) \mathbf{w} + \left(\frac{Y_T}{qB_T} \right) \Sigma^{-1}_T \left[\Sigma^X_T + \zeta_T \left(\frac{q^{-1}}{1 - a^{rf}_T} \Sigma^a_T - q\Gamma \Sigma^A_T \right) \right] \mathbf{w}.$$
$$- \chi_T \Sigma^{-1}_T \Lambda_T \left[\left(\frac{1}{\xi_T} - \phi \right) \left(\boldsymbol{\omega}_T - \hat{\boldsymbol{\omega}}_{T-1} \right) \right].$$

□